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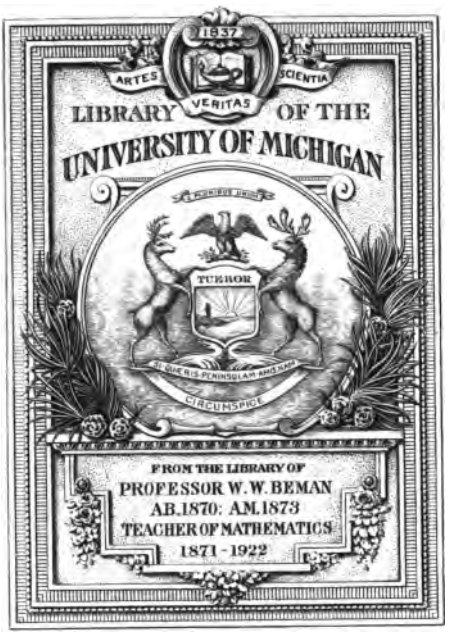
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MODERN ARITHMETIC METHODS AND PROBLEMS

By

THEODORE LINDQUIST

KANSAS STATE NORMAL SCHOOL

EDITED BY

GEORGE W. MYERS

THE UNIVERSITY OF CHICAGO

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AUTHOR'S PREFACE

Students below senior college rank receive much greater benefit from a class in which instruction is based upon a good text than upon lectures. The great majority of students in classes for teachers of arithmetic need to consider subject-matter as well as methods of presentation. For this purpose the old form drill-book which gave a review of grade arithmetic is inadequate, as is also the purely methods-book which discusses methods only to the exclusion of all subject-matter.

It is to supply the needs of the average teacher of arithmetic that the present volume has been evolved by the author while actively engaged in the instruction of such teachers. During the past two years this material has been used with excellent results in syllabus form. The plans and the suggestions herein offered are such as have been found productive of the best results by teachers in actual practice under varied conditions.

The treatment contained in the text derives its plans and methods from the interests, attitudes, and aptitudes of the children. It looks upon sane motivation and socialization of arithmetic as of the highest importance; it makes economy and real efficiency the ideals and the touchstone of good method; it demands a broad and sound scholarship as basal to all efficient methodology; it makes the search for the "**big idea,**" the "**big thing,**" always the

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first concern of the teacher. As most young teachers need a review of subject-matter, it demands all along a thorough reviewing, recasting, and reorganizing of subject-matter, seeking always to train the prospective teacher to become an effective thinker. A *sine qua non* in teaching children how to study is that the teacher be an example of a student.

The development presented in this book is the outcome of considerable varied classroom experience. The form in which material is put before beginning teachers has been found of greater importance than it would appear to be. Small units of work that can be given a sort of completeness in a short period of time are more fully grasped and better retained than are longer and less broken unities. The short chapter has been found the more serviceable form for the structure of the material used, and it has accordingly been adopted in this text.

The subject-matter contained in the exercises at the end of each chapter has been selected with a twofold object: to serve as a review of subject-matter and also to suggest live material with which to supplement the texts the teacher will use later. Prospective teachers needing a review of subject-matter will study the exercises from both standpoints, while experienced teachers not needing this work will study them from the second standpoint only. This text will be found adapted to a one-semester course of from two to five hours, depending upon how much time is given to a review of subject-matter.

Appended to each chapter will be found a short bibliography which has been selected for the accessibility of the references as well as for their bearing upon the topics treated in the chapter.

The book is suggested as a text for experienced or inexperienced teachers in college, normal, or high-school

training courses; teachers' institutes; reading circles; individual self-help; and correspondence courses.

It is my privilege here to acknowledge the many helpful suggestions and criticisms which have come from my colleagues, Miss Lena B. Hansen, Mr. W. H. Keller, and Miss Avice R. Wright; Dr. Herbert G. Lull, of the Kansas State Normal, who contributed the problem and a large part of Chapter V; and Miss Flora J. Cooke, of the Francis W. Parker School, who contributed the material for Chapters IX and X.

EMPORIA, KANSAS. THEODORE LINDQUIST.
October, 1917.

EDITOR'S PREFACE

The characteristic notes of modern arithmetical education, as indeed of all modern education, are practicality, social worth, and psychological soundness. The oft-cited ideals of economy and efficiency are not specifically mentioned because they are implied in the broader ideals of practicality and social worth. The treatment of arithmetical methods and problems of the following pages claims to be in accord with these characterizing notes.

Practicality Practicality in educational material implies a high degree of teachability, close similarity to the actual, and a ready usableness of results. In educational procedure it means:

1. Instructing children in the fund of arithmetical knowledge and training them into the measure of arithmetical skills that the race has judged worth preserving for them.
2. Giving children some preparation for the occupations and vocations of modern adulthood, and
3. Doing all this by methods that will foster the pupil's native enthusiasm for learning.

Teachability may be gauged either empirically, by classroom experiment, or objectively by intrinsic characteristics of the subject-matter. Let it be noted in passing that the material of this book has grown into its present form under the shaping influence of extended classroom trial,

the final form, herewith offered, having been attended with marked success.

Attention is directed to the following internal marks of teachability:

The book makes simplicity, childlikeness, the guide to presentation in the early grades and usability the guide in the later work. It struggles continually for motivated, and purposeful work as the pupil gauges motive and purpose. It seeks to habituate the teacher-pupil to search in every situation for the essential, the controlling, the "big" idea, and to implant this habit into his pupils. This is the first great step in acquiring the secret of how to study.

The book attempts to judge accurately and to exemplify fully the stages at which to discontinue topics with children. It supplies an abundance of well-graded practice material of the right sort in the exercise lists appended to the chapters. The bibliographies furnish ample sources for supplementary reading and study on the topics treated. These teaching aids guarantee teachableness to a high degree and go far toward realizing the three meanings of practicality stated above.

After making it plain to the pupil-teacher that considerable systematic drill is essential—that the "big" thing is to have the drill exercise snappy and stimulating—problem material drawn from real life situations is prescribed, and exemplified, and copious sources of such material are cited.

The text draws so fully from social, industrial, and commercial sources both material and guidance for the way of treating this material as to give it a close approach to actuality. Sometimes a better way than the one actually used is shown, teachers are even counseled to try to get men of affairs to adopt the better way, but the habitual

way of the workaday world is always sufficiently well treated and exemplified.

Such material, so presented, with the degree of scholarship urged and fostered by the text, must beget in teacher-pupils the *habit of use,* both as to results and as to methods of study. This usableness will perforce carry over into the working habits of the children who are so fortunate as to have such teaching.

The editor feels then that whether judged empirically or from internal merits, the book must be conceded to possess marks of high practicality.

Social Worth Enough has already been said as to both material and method of this treatment to prove it of high social worth. There is, moreover, an excellent chapter treating *socialization and motivation* explicitly. In this chapter the social significance of arithmetic is fully brought out and exemplified. Its values speak for themselves.

A phase of socialization of arithmetic that is not generally well attended to, is the employment of arithmetical material from the other school subjects. The school as a social group has certain common interests and concerns by reason of the school subjects that all are pursuing. It is distinctly poor practice not to make use of this opportunity to secure the social value of this group interest. The author of this book has done well to consider this matter, and to show how to make this socializing phase profitable.

Psychological Soundness A psychological treatment presents matter through its *function,* its office. It teaches how to do by actual doing. It modifies and adapts adult knowledge to immaturity. It takes its point of view and its departure from what the

child can and will do. It makes teaching-technique such that it becomes study-technique to the pupil. The psychological way of the teacher, rightly conceived, is always the normal way of learning to the pupil. These ear-marks of psychological soundness will be found again and again by the careful reader, not in a few isolated places, but throughout this book.

The editor, therefore, claims that this volume meets the three essential characteristics of modern education, viz.: practicality, social worth, and psychological soundness. It may then with all propriety be termed a modern book. He feels, moreover, that this modern book for arithmetic teachers will aid in reducing the number of those who merely take a clerical attitude toward past education, and will augment that rapidly growing class who take a progressive attitude toward present education. We need more books capable of this function.

GEORGE W. MYERS.

Chicago, October, 1917.

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MODERN ARITHMETIC METHODS AND PROBLEMS

CHAPTER I

INTRODUCTION

1. Poor Teaching of Arithmetic. Efficiency, service, results; these are the demands of all employers in the commercial and industrial world of today. More and more, this is becoming the slogan among those who employ teachers, as well. Yet in the face of these demands, there is still a great deal of very mediocre teaching throughout the country. Especially is this mediocrity noticeable in arithmetic. Comparisons of the work done in arithmetic with that done in other subjects have, as a rule, been far from complimentary to arithmetic. For example, it has been found by actual test, that the range in the abilities of children to spell varies only slightly in different schools. In arithmetic, according to the same test, the ability varied from 8.9% to 81.1% in the 7th grade; and from 11.3% to 91.7% in the 8th grade; while the range of ability in schools taken as a whole averaged from about 25% to 80%.*

The severest criticism, however, concerning the lack of efficiency on the part of the pupils who have completed the grade work in arithmetic comes from business men who

* J. M. Rice, Educational Research: A Test in Arithmetic. Forum, 34: 281.

employ these young people and from the teachers of the high schools. Some of these criticisms are, of course, mere fault-finding, but a great many are entirely warranted. This lack of ability on the part of the young person to apply to general practice the arithmetical principles which he is supposed to have mastered in school is the more conspicuous and the more subject to criticism in comparison with the other subjects because of the practical importance of the subject itself, and its inseparable relation to the business world.

The reason for this criticism must therefore be twofold: first, the teaching of arithmetic in the grades is often inefficient; second, the important part that this branch plays in all business life brings this inefficiency into the lime-light, as it does not do in the case of the other subjects. Notwithstanding, many teachers and superintendents go unconcernedly on their way, paying no attention to these criticisms and making no attempt to improve that portion of arithmetic teaching for which they are responsible. Or, if they do stop to give it a momentary consideration, they attribute the lack of efficiency to the limited time allotted to the subject and, therefore, consider it an evil that can not be remedied.

On the other hand, it was found by a late investigation * that 15.25% of the total school time in fifty large cities of the United States was devoted to arithmetic. While this is by no means an exorbitant amount of time to give such an important subject as arithmetic, it is, at the same time, entirely sufficient to produce good results. Many instances may indeed be cited where satisfactory results have been obtained in less time.

* International Commission on the Teaching of Mathematics. American Report on Mathematics in the Elementary Schools of the United States. United States Bureau of Education. Bulletin, 1911. No. 13: 17.

Therefore, it behooves every one connected with the grades, from the primary teacher to the superintendent, to concern himself in removing the cause for this stigma put upon the grade teacher of arithmetic, by making a personal effort to raise the status of the arithmetic work to a point commensurate with its importance. It is for the purpose of aiding in clearing up this situation, and especially of helping the inexperienced teacher, that this little volume has been prepared.

2. Why Arithmetic Is Taught. Arithmetic is the quantitative side of the grade curriculum. It deals with measurements, with size, and with comparisons of measurements and of quantities. Arithmetic answers the questions, "how many?" and "how much?" Without this attention to the quantitative side of the curriculum, the other subjects of the grades would lack an essential element, and some of these subjects, such as music, would be wholly eliminated. The games of children would be much more crude and our general civilization would be that of the savage whose number-knowledge is limited to, "One, two, three, heap many." Our daily lives are full of these quantitative questions involving number relations and measurements, as to "how much" and "how many." It is not sufficient to know *what* medicine to use, *what* ingredients to put into a cake, or *that* a savings bank pays interest. The necessary thing is to know the *amount* in each case. From the simplest child's game, then, to the building of the most complicated modern machinery, counting and computations enter as a necessary element.

3. The Utilitarian Value of Arithmetic. Much has been said and written of late regarding the utilitarian value of arithmetic,—the "bread and butter" side of its importance. The need of a good command of arithmetic by the skilled mechanic, by the up-to-date farmer, by the progressive pro-

fessional man, by the successful merchant, and by the efficient housewife, is so obvious as to need no discussion. Much has also been said regarding the amount of arithmetic necessary for conducting these practical affairs of life. This will, however, depend upon the vocation followed; but more upon how far the individual progresses in his vocation.

The ability to compute is not the only utilitarian value to be found in the study of arithmetic. Life is one long series of problem-solvings. Situations are met with daily that call for an exercise of judgment. Conclusions must be drawn from given data. This exercise of judgment receives its first training in the solution of the first little problem in the earliest grades and continues in a greater degree through all the subsequent grades. If the pupils, when they have passed through the eight grades, are unable to draw such inferences properly, the teacher, not the arithmetic, must receive the blame.

4. Arithmetic for the Larger Life. Just because arithmetic has an essentially practical value is no reason why it should be studied solely for that reason. A child is not taught music that he may become a great musician, or drawing that he may become a great painter and thereby receive big financial returns. If English in the schools were to be restricted only to teaching that which is necessary to carry on conversation with the butcher, the baker, or the candlestick-maker, as is sometimes argued concerning the extent to which arithmetic should be taught, then English in the grades might be abolished altogether. A child can *pick up* enough English to "get along" through life. But we can not force ourselves to believe that the material aid teachers render is the only function of the American schools. It is true that they must first of all equip their pupils as bread-winners, but they must also prepare these pupils for the larger and more complete life; that which

takes cognizance of the esthetic and the ethical as well as of the material.

5. The Doctrine of Formal Discipline as to Arithmetic. Space will not permit a discussion concerning the validity or invalidity of the Doctrine of Formal Discipline; namely, that training in the study of a particular subject will aid in the comprehension of other subjects and in the general development of mental brawn. Such conclusions as are necessary to establish the proof or disproof of any theory can be established only after a long time, during which a large number of varying instances have been studied. Before long very decided modifications in the matter of the rejection of this theory may be looked for. It must first be mentioned that many of the claims made for development through the study of arithmetic are not made for it by its most ardent supporters. It can not be argued that a girl developing keenness and insight in her arithmetic will have her keenness in the discriminating use of seasoning in cooking developed. Her arithmetical keenness will, however, serve her in modifying the size of recipes, in deciding whether to buy food in large or small quantities and so on. No one can dispute that arithmetic possesses a valuable mode of thought found in no other subject of the grade curriculum and also that it offers an opportunity for the exercise of judgment to a degree afforded by no other subject.

Arithmetic has a definiteness and the results obtained in it have an absoluteness which makes it a study unsurpassed in the mental training it affords. In some subjects a pupil can skim over a lesson and yet fool either himself or his teacher that he has accomplished the desired results. This he is unable to do with a lesson of arithmetical problems, because for each he must secure a definite result which is either right or wrong. It has an ethical value as well,

which is too little emphasized. The lesson of absolute truth and the necessity of accepting nothing short of this truth is a most valuable one. Moreover, the opportunity it gives of teaching the need of care, neatness, thoroughness, system, accuracy, industry, simplicity, perception, and keenness in seeing below the surface, is not lightly to be passed over. If the children go through the grade schools without receiving the above-mentioned benefits from their study of arithmetic it must be borne in mind that the opportunities have been there and that either the children have opposed these advantages too stubbornly, or, what is far the more likely, the teacher has failed to take advantage of his opportunities to inculcate these habits.

Arithmetic is not merely a study of numbers,—of two and two, and four and four. It is a study of proportions, of relative values, of keenness in discrimination, of foresight, and of arriving at conclusions. Breathe into it these qualities and you will breathe into it the oxygen necessary to make it a living, vital subject. When you have done this, watch the difference in your arithmetic class. Instead of dull acquiescence and phlegmatic inertia it will be alive to the finger tips.

6. Improvement in Arithmetic-Teaching. Mere methods furnish no panacea for the difficulties of teaching, and can never take the place of scholarship or of common sense; nevertheless, to obtain the best results, methods must be given vital consideration. At the same time, the teacher is cautioned about being led away by this or that new method which is to rob the teaching of arithmetic of its perplexities, and make it a joy forever. Always be rational; never be an extremist or a faddist.

7. A Broad Knowledge of Subject-Matter Indispensable. A comprehensive knowledge of the subject-matter is a prime requisite of good arithmetic teaching. It would be as absurd

for one who has never played the piano to expect to teach piano, as for one lacking a knowledge of arithmetic to try to teach that subject. The successful teacher of any portion of arithmetic must have a grasp of the whole field of the subject. Not only this, but to be successful in the biggest sense of the word he must supplement this thorough knowledge of arithmetic by at least a year of algebra and a year of plane geometry. Only by such preparation can he adequately comprehend and impart the mathematical principles which have their root in arithmetic.

8. Seek Always for the "Big Idea." This thorough knowledge of the subject-matter should be followed by a workable course in methods of teaching. In this course, as well as during the later work of teaching, the prospective teacher is urged to be on the lookout for usable ideas. These ideas may later be modified to be adaptable in the special cases where they are needed. The student is also advised to be constantly on the alert for the **big idea** so that the essentials are not confused with the incidental details. The latter must never be accepted as the distinguishing features of a method. For instance, it is immaterial whether drills come at the beginning, at the end, or during the recitation period, or even at some other time of the day. The **big idea** is to have all such drills snappy and full of life.

The successful teacher is ever trying out new ideas. This form of experimentation, if carried on in a sane, quiet manner, by easy stages that will produce no revolutionary results or cause bewilderment in the minds of the pupils, cannot be harmful but will, on the contrary, help the teacher constantly to improve and grow in his work. "Evolution not revolution" must characterize all such experiments with children. Whenever possible the teacher should compare present with past results. Above all it behooves the future teacher to get into the spirit of the work first,

while a student, and to grow in spirit and enthusiasm during the later work of teaching.

9. Use of the Following Chapters. The first eight chapters of this book are of a general nature; each of the remaining ones deals with some topical phase of arithmetic. No reference will be made to the subject-matter except in the exercises which contain a few problems for review. These are to be used, primarily, for acquainting the student-teacher with what constitutes good problems. Any good arithmetic text is recommended for consultation. Suggestions and discussions which it may be desirable to preserve may be kept in loose-leaf notebooks and should always be written in ink. Legal size envelopes will be found useful for preserving paper-clippings and similar material. To make this work of inspirational value, let the student endeavor to pick out the essentials from the incidentals in each chapter and to grasp the **big idea** contained in each. This care will aid him later in placing the proper emphasis while teaching. Try to find in each chapter as here presented some idea for future use in the schoolroom, and as far as possible illustrate each point brought up for discussion. Nothing else will be of greater aid in elucidating and in fixing in the mind the matter under consideration.

I. EXERCISES

1. What is the present status of arithmetic teaching in the United States? Wherein lies the cause? How can it be remedied?
2. How much time should be devoted each day to arithmetic in the grades?
3. State the essential differences between arithmetic and the other subjects of the grades.
4. Discuss arithmetic as illustrating the quantitative side of the grade curriculum and of life.
5. Discuss the value of arithmetic from the "bread and butter" standpoint.

6. Discuss the value of arithmetic aside from its value in gaining a livelihood.

7. What is meant by the Doctrine of Formal Discipline? Discuss the meaning of its overthrow.

8. How does this doctrine apply to arithmetic?

9. Discuss the necessity of wide knowledge of the general field of arithmetic and of the methods of teaching arithmetic.

10. Explain what is meant by the "big idea." Give two illustrations.

11. Distinguish between training and education. How may both be derived from this course?

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CHAPTER II

EARLY HISTORY—NOTATION AND NUMERATION

10. Why Study the History. Any science becomes more real, more vital, and its value more apparent through the study of its history. Such a study appeals especially to the teacher since the individual develops in a manner quite similar to that of the race. Therefore, two chapters will be devoted to a consideration of the evolution of our number-system and the general development of arithmetic. The history of the development of the various separate topics will be taken up in connection with the teaching of each.

Of necessity these historical references must be very brief. Hence it will be impossible to enter into a discussion of any of the many mooted questions which are engaging the attention of historians. Those interested in a more complete development of any historical phase of the subject are advised to consult further some of the histories mentioned in the appended list, p. 17.

11. Primitive Counting and Number-Reckoning. The ability to count, to a limited extent, is an accomplishment of the various tribes of today that possess only the lowest form of civilization. This ability may be limited to the counting of only one, two, three, four, five, and *many;* but it is counting, nevertheless. Most of this counting is done on the fingers and in many tribes two men are needed to count beyond ten. Early merchants communicated prices to each other secretly by the touching of hands under their clothes; today men on the board-of-trade signal to each

other by means of a finger language; even the numerals of the universal Hindu-Arabic number-system are called *digits*. Further, except in two cases, the *radix*, or base, has been found to be five, ten, or twenty, showing quite conclusively that the earliest form of counting was founded upon finger-reckoning. Hence the origin of ten as the base of our number-system is not a great mystery.

12. Numbers to Different Bases. It is quite fortunate that ten was selected in the place of five or twenty. With five as a base our present number-system would consist of only the symbols 1, 2, 3, 4, and 0. Two figures, or 10, would then be required to express five. The symbols 6, 7, 8, and 9 being absent, the numbers they represent would be written 11, 12, 13, and 14, while the symbol for ten would be 20. Twenty-five, which is five fives, would then be expressed by 100 just as we now express ten tens. One hundred twenty-five, twenty-five fives, would be expressed by 1000, which we now use to state one hundred tens. Just as five is too small for a convenient number base, twenty is too large. Its use would necessitate ten additional symbols for 10, 11, etc., through 19, and two figures, or 10, would be used to express 20. Twenty twenties, or four hundred, would be expressed by 100. It has long been recognized that it would have been even more fortunate if twelve had been selected as a base, but so also has the futility of trying to bring about a change after all these ages.

13. Tallying and Other Forms of Counting. Pebbles, sticks, nuts, notches cut in sticks, and similar crude instruments were used in the early counting and recording of numbers. Finger-counting was developed into a regular system. One man counted the units, a second the tens, a third the hundreds, and so on. Accounts were kept by notches cut upon a stick which was split in order to give a record to each of the interested parties. From this we

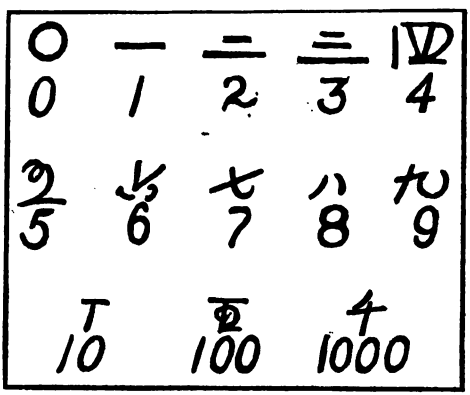
have the expression "keeping a tally." Later forms are counting upon the abacus and writing in sand strewn upon stone tablets. The only permanent records of those early times were inscriptions upon stone tablets and coins. Counting and recording numbers were thus carried on extensively before the invention of our present number-system.

14. Hindu-Arabic Number-System. The Hindu-Arabic numerals compose the arithmetical alphabet. These number-symbols are used to form numbers just as the letters of the more common alphabet are used to form words. Like the letters, the number-symbols have greatly changed in form.

15. The Two "Big Things" in the Hindu-Arabic System. The symbols of the Hindu-Arabic system are not especially simple; V, of the Roman notation, is fully as simple as 5 while C and M are made more easily than 100 and 1000. The first **big thing** of this system is the idea of place value which gives a digit ten- or a hundred-fold value in one position over what it would have in another. Until the coming of the Hindu-Arabic system a special character was used to express each number, or it took the value of several characters added together so that forty was expressed by 𝍠 and fifty by 𝍡, in which 𝍢 represents twenty and 𝍣, ten. Another system based upon this *additive principle* is the well known Roman system. The Chinese and Japanese are today breaking away from such a system. They have characters for the zero and first nine numerals together with separate ones for ten, hundred, thousand, and so on, a few of which are shown on the opposite page.

They write forty with the symbol for ten and that for four beneath it; five hundred with the symbol for hundred and five beneath it, and so on. All such number-systems are of course nearly useless for computation purposes. The idea of place value had its origin in words written one after the other about the sixth century A. D. The second **big**

thing was the invention of the zero a century or two later. These two ideas practically constitute our present simple number-system for integers. Before the invention of the zero, numbers could be expressed and computations carried out by means of the *abacus* in which the zero was not needed. The term, million *(great thousand)* was introduced by the Italians in the fourteenth century and our present numeration was completed about 1520 by NICOLAS CHUQUET, of Lyons, France, who added the terms billion, trillion, and



CHINESE NUMERALS

so on. The present simple number-system has been in universal use for only about four hundred years. The science of arithmetic had previously been retarded for a long time because of the lack of a simple and usable number-system.

16. The Hindu-Arabic, the World's Counting System. The universal adoption of the Hindu-Arabic number-system —it is now being introduced into both China and Japan— makes it the great world counting-system. Variations found in the different countries are very slight. In Germany the comma is used for the decimal point while in England

it is the dot placed higher up than in the United States, thus, 2·5. In southern Europe and in the United States large numbers are separated into periods of three digits, while in England and northern Europe they are separated into periods of six digits so that 23,567,486,356,892,139,504 would be read

23, 567486, 356892, 139504
trillion—billion—million—same as in United States.

Only one character, the 9, shows any marked difference in the printed and written forms. Such a world-wide adoption of this number-system speaks volumes for its simplicity and adaptability to universal needs.

17. The Origin and Distribution of Hindu-Arabic System. The Hindu astronomers considered themselves experts in astronomy and began this simple number-system to aid in carrying out their computations. As is often the case, the secondary purpose wholly eclipsed the primary one. The numerals soon passed into the hands of the merchant class by whom they were further developed. Because the merchant class was not the writing or learned class, very meager records were left for later historians with which to trace the development of the symbols extending over a period of several hundred years. The fact that the educated class did not develop the system also retarded its progress very much.

As the Arabs, through whom knowledge of the times was disseminated, came into contact with the merchants of India, they naturally adopted the simple number-system of the latter and spread it to other parts of the civilized world. The Arabs also wrote treatises on this simple notation. From this fact the idea gained currency that the notation was of Arabic origin, although the early Arabic writers very properly acknowledged the invention as coming from

the Hindus. The Arabs were the distributers and in no sense the inventors of the system. The Hindu numerals were used very extensively by the Italian merchants during the thirteenth century, but hardly at all by those of northern Europe before the middle of the fifteenth century.

18. Notation and Numeration. As separate chapters will be devoted to the early number work of children, only a few general suggestions will here be given as to counting, and the reading and writing of numbers. Let the child's number development come as a need in his growth; as a necessary part of his games. Some of the best work along this line is done incidentally, or episodically. The teacher, however, must not confuse *incidental* with *accidental*, and expect this development to happen of its own accord without a directing hand.

The teacher must ever bear in mind that this development on the part of the child follows closely that of the race. The idea of place value proved difficult to the race and is in like manner a stumblingblock to the child. Much attention must, therefore, be given to this matter in order to bring it home to the child. Sticks, as toothpicks, make excellent early illustrative material to show the ideas of tens and hundreds. A number as 444, in which each figure is written in differently colored chalk, can be used to advantage in explaining place value in the written number. United States money will also be found helpful in teaching place value. Numbers containing zero always give difficulty and hence must be taken up slowly and thoroughly, among which must be given especial consideration such forms as 20, 200, 304, 460, 7000, 6800, 4060, 3007, etc.

The teacher is cautioned against destroying the child's initiative by telling him too much directly. For instance, after the child has been told the numbers through fifteen, he should be permitted to work out the others of the "teens"

by himself. The same procedure is recommended for the twenties, thirties, and so on; also for the hundreds. A child's reasoning can best be developed by stimulating him to exercise it. And in no other study is there such opportunity for helping the child to a cultivation of the reasoning process as in arithmetic. Attention must also be given to the writing of numbers in words—including spelling—as forty, thirty-five, two hundred fifty-six, and so on.

19. Roman Notation Being Superseded. The Roman notation is more and more being replaced by the Arabic and at present is found mainly in chapter headings and on some clocks. It need not, therefore, be given special attention.

Number-symbols and letters arranged one after the other in such a manner as to spell out words, after the number-symbols have been translated into letters through the Roman notation, will be found a stimulus in teaching this notation. Thus IOOOW becomes COW, 5OION becomes LION, and so on.

II. EXERCISES

1. Why should teachers study the history of any subject they are to teach?

2. Describe the early methods of counting and recording numbers.

3. What is the base or *radix* of a number-system? What is it for the Hindu-Arabic system? What is the most likely origin of this system?

4. Suggest a better base. Why is this better?

5. Make the following addition using 5 as a base: $313 + 241 + 142 + 401$. Make the following multiplication using 5 as a base: 231×14.

6. Why are the Hindu numerals called the "Arithmetical alphabet"? Compare its universality with that of the literal alphabet.

7. What are the two big things in the Hindu number-system? Illustrate.

8. Compare the Roman and the Hindu number-systems.

9. Give a brief history of the Hindu number-system. Why is it called Arabic?

10. Explain the teaching of place value to children.

11. What numbers are likely to prove troublesome? How can this be remedied?

12. What error must be guarded against in teaching counting?

13. Suggest additional puzzles to be solved by the use of the Roman notation.

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CHAPTER III

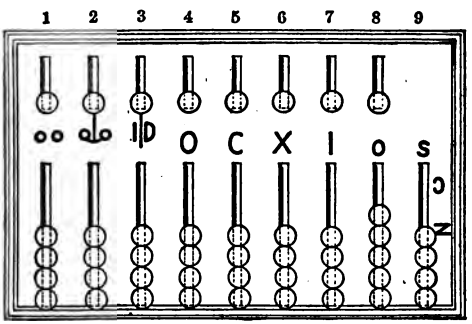
THE DEVELOPMENT OF ARITHMETIC

20. Early History of Arithmetic in Europe. Prior to the invention of the Hindu number-system, progress in arithmetic was extremely slow owing to the great labor required in making computations. For several centuries after the adoption of the new number-system the development was far from satisfactory due to the fact that the merchant, and not the learned class, was mainly interested in computations and arithmetical matters. Wherever those of the learned class became interested in arithmetic, as they did in Greece, they devoted themselves to the philosophical side wholly; hence they did not aid in the development of either the science or the art to any great extent.

21. How Hindu Arithmetic Got into the Schools. Computation with the abacus was general from the time that the idea of place value became universally known in the civilized world until the writing of numbers with the zero-symbol was finally adopted, some time from 400 to 1100 A. D. The abacus is still used extensively in China and Japan. Before the invention of the zero a number like 407 could not be written but could be expressed on the abacus by merely leaving vacant the second column, that for 10's. Like many another antiquated custom the abacus persisted long after its usefulness had passed away with the invention of the zero. The complete Hindu number-system containing the zero was first used in computation with figures written in dust strewn upon small stone tablets. The school of

men using these dust *(gubar* or *gobar)* numerals was called *algorithmic,* from the name of an Arabic writer on algebra. For several centuries these two schools of arithmetic contended for supremacy until the *abacists* had to give way to the simpler and more scientific method of the *algorists.* Widely used almanacs and astrological tables aided in establishing the algoristic numerals.

It was a common thing during this period of rivalry between the two methods to have cities legislate against the use of these new numerals in business records. The church



AN EARLY FORM OF THE ABACUS

schools and monasteries did very little at this time to promote arithmetical knowledge, as their only interest in computation lay in calculating the correct time of the yearly coming of Easter.

With the revival of learning, the higher institutions took up the study; this was productive of excellent results. A more potent influence, however, was felt in the merchants' organizations, which recognized the value of arithmetic, and they encouraged the establishment of counting schools in which arithmetic and accounting were the principal subjects taught. These special schools for teach-

ing counting prevailed until about a century ago when arithmetic was introduced more extensively into the grades. The work included in the grades has been increased by easy steps until the present conditions prevail.

22. The Influence of Early upon Later Arithmetic. The early arithmetics contained many complicated puzzle-problems designed, it was declared, to "quicken the mind." The later works written for the commercial schools contained much material useful to those studying it at that time as complicated partnerships, fractions with large terms, involved money exchange, reduction of weights and measures from one table to another, various rules of interest, etc., much of which persisted in the later books long after it had become obsolete through changes in commercial activities. The books written about 1000 A. D. were largely treatises on the use of the abacus, which later gave way to the algorithmic writers and their dust numerals. The first English arithmetic was issued by CUTHBERT TONSTALL in 1522 and was written in Latin.

The following are some of the men connected most prominently with the development of early arithmetic: GERBERT (Pope Sylvester II), BOETHIUS, LEONARDO OF PISA (Fibonnaci), TARTAGLIA (Nicolo Fontana), CHRISTOFF RUDOLFF, ADAM RIESE, SIMON STEVIN, JOHN NAPIER, PESTALOZZI, TILLICH, and GRUBE.

23. Arithmetic in the United States. During the colonial period and the first years after the adoption of the constitution, very little arithmetic was taught in the United States, and that poorly. Although an arithmetic was published in Boston as early as 1729 and PIKE'S *Complete Arithmetic* appeared in 1788 the books used were mostly English publications. At first, texts were very scarce and it was a rare incident to find two copies of the same text in any one school. The teacher did not conduct recitations

in arithmetic, but "set examples" for his pupils to work out individually. These examples he selected from his private manuscript-book containing not only the examples, but the complete solutions as well, which it was the duty of the pupil to reproduce. WARREN COLBURN published his *Intellectual Arithmetic* in 1821, which was farther ahead of its time than any text has been that has appeared since. His success was no doubt due in large measure to the fact that he took account of the natural impulses and desires of the normal child in the selection of his problem material. According to his statement, the children he taught guided him in writing his book.

Constantly increasing attention was given to arithmetic throughout the years, until in the latter part of the nineteenth century it had become one of the chief subjects of the grammar schools. Rules were formulated as the basis of the texts; topics were rigidly separated from each other; and the problems were often long, complicated puzzles only slightly related to existing commercial conditions, or the life of the child. Unfortunately this state of affairs is still quite prevalent in our present day texts. Special methods have arisen from time to time which were offered as panaceas for arithmetical ills but unfortunately they have not been panaceas. Some of these will be considered under separate paragraphs.

24. Topical System. The teaching of arithmetic by some form of the topical plan has been used from the very first. In its extreme application any topic such as fractions, denominate numbers, or interest, is considered exhaustively at one time, when it is studied for a considerable period and is treated as comprehensively as the understanding of the pupils will permit. As soon as a so-called topic is "completed" reference to it thereafter will be slight and wholly accidental. Such a plan excludes a mature treatment of

any subject taken up during the early years, and also excludes, during these years, the use of any topic material which is deferred to the later years. It is well to hold to the topical plan sufficiently to conserve and concentrate the energies of the pupils and give methodical direction to their study, but certainly not in the extreme measure heretofore mentioned.

25. Spiral System. The spiral system is just the opposite of the topical system, in that each topic is studied several times during the eight years of grade work. For instance, mensuration is considered in the very first grade in the measurement of lengths, next in finding areas of rectangles and volumes of rectangular solids, and in the later grades in finding the areas of surfaces and volumes of various geometrical figures, together with their construction. Each time a topic, or principle, is brought up the numbers used are larger, or more complicated, the problems are more difficult, and the conditions more general.

GRUBE is more widely known in connection with the spiral system than any other one man. He had very radical views on the subject and insisted that each number be known in its entirety before the next number should be taken up. For instance, he believed that all the various factors, combinations, and operations for any one number such as 12, should be thoroughly comprehended by the pupil before 13 could be considered. The work was confined to numbers up to 10 during the first year and up to 100 during the second year. This theory was given a thorough trial in Germany, but was found unsatisfactory. Several texts, based more or less extremely upon the spiral system, have appeared from time to time. In the most extreme of these, a topic will be found treated regularly every fifty or sixty pages. Such a method of rotation will be found far too mechanical to secure good results. The propriety of calling into use a

principle or operation previously learned, whenever occasion arises, is, however, clearly recognized. So a rational combination of these two plans which most texts of today are trying to embody, will no doubt prove by far the most satisfactory procedure.

26. Object Teaching. Pestalozzi was really the originator of object teaching, as we know it today. He was not an extremist and used objects only to make his teaching the more real and appealing to his pupils. Tillich, a pupil of Pestalozzi, carried object teaching much further, and SPEER, of this country, carried it as far as might be expected of such an ardent extremist. It was found, however, that pupils who had used the Speer method were less advanced than those not using it; and hence, it has been almost universally discarded.

27. Two Views of Object Teaching. There are really two ways of looking at this matter of object teaching; one is to base the idea of numbers upon concrete objects, so to speak, and a second to make use of objects, such as blocks, measuring, drawings, paper-folding, paper-cutting, and so on, for the purpose of motivating and clarifying the work in hand. As noted, the first has not been productive of good results and the teacher is hence warned against such a continued use of one object or illustration that the children will connect an idea permanently with that one object. The value of the second use of objects throughout the work in arithmetic, such as drawings in the solution of problems, can hardly be over-estimated. In addition to drawing to scale and graphic representation, which will be considered in detail later, there is the use of sketches in picturing the conditions of a problem to be solved. Difficult problems are often made simple by this means.

The discontinuance of the use of objects as soon as they are no longer necessary is urged just as strongly. A crutch

is not meant to be used after the need for it has passed away. Many teachers hold that objective work is to be gradually discontinued with each advancing grade. This is a most erroneous idea. Each new topic, as business transactions and mensuration in the last grades, needs to be introduced fully as objectively as counting in the very first grade. Such a use of objects will insure arithmetic instruction based upon the fundamental principle of proceeding from the concrete to the abstract.

28. Rational Method. The earliest arithmetic work was based upon arbitrary rules laid down by the teacher and the text; later the teacher essayed to explain these rules; and at present we endeavor to stimulate the children to formulate rules and methods of procedure for themselves. As an illustration may be mentioned the development of the formula for finding the area of a triangle. Each pupil arrives at the same result working with a different model, thus establishing for himself the truthfulness of the formula arrived at. This is a method which has very much to commend it, but it is one in which there is grave danger of overdoing, and it requires a great deal of discrimination on the part of the teacher.

It is necessary for the teacher at all times to inquire into how far any part of the work should be studied *as an art* and how far *as a science;* how far to carry the *ability to do,* and how far the *ability to understand* the process; how far to *habituate,* and how far to *rationalize.* The extent to which rationalization may be carried will depend largely upon the individual members of any particular class. It is too much to expect that *all* of the pupils will grasp *all* of the reasoning underlying the processes *all* of the time. It is a most decided gain to have the majority of the class do so. Again, much more can of course be done with the older than with the younger pupils.

Upon one thing the teacher must insist—that every pupil be able to *do the work* even though the reason underlying it be not fully mastered. It is also very essential that every pupil be required to do this work for himself and that he does not remain a mere passive listener to the teacher's explanations. A most useful plan for the teacher to follow is to increase the difficulties gradually, to go from the concrete to the more abstract, and to encourage the use of the same ordinary common sense which the pupils would bring into action when solving some difficulty arising upon the playground, or in some other activity. The progressive teacher will make use of the above modern ideas; the unprogressive teacher will teach by rule and the unproductive explanation.

III. EXERCISES

1. Discuss the development of arithmetic in the early centuries after the invention of the Hindu number-system.

2. Describe the abacus. Why was it used in the early years of the Hindu number-system?

3. Discuss the development of arithmetic after the coming of the gubar numerals.

4. Discuss briefly the early arithmetic texts. Compare with those of today.

5. Give brief biographical sketches of the men mentioned in sec. 22. *

6. Give a brief discussion of early arithmetic teaching in the United States.

7. Give a brief biographical sketch of Warren Colburn. What was the nature of his arithmetic?

8. Discuss the teaching of arithmetic by the topical plan; its advantages and its disadvantages.

9. Discuss the teaching of arithmetic by the spiral plan; its advantages and its faults.

10. Compare the topical and the spiral plans. What modifications seem advisable?

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*All references are to section numbers.

11. Distinguish between the two forms of object teaching mentioned.

12. Discuss the advantages and dangers arising in object teaching.

13. Illustrate the use of objects in presentation to a class.

14. A farm of 320 acres is four times as long as it is wide. Show by a drawing how simply its dimensions can be ascertained.

15. A glass cylinder of 4″ diameter contains 6″ of water. If the water is raised to 7⅜″ by immersing an irregular lump of lead into it, what is the volume of the lump of lead? Illustrate with a drawing.

16. John and Henry start from A at the same time on bicycles, riding in opposite directions at the rate of 12 and 15 miles each per hour. How far apart will they be in 3 hours? In 5 hours? Solve by the use of a drawing.

17. Suggest at least four other problems suitable for class work similar to the three above in which drawings or objects will aid in giving a clear solution.

18. Discuss the rational method of teaching; its advantages and its dangers.

19. How far should the work be habituated and how far rationalized for the various grades? Illustrate.

20. Explain fully how a subject should be introduced inductively by the use of a concrete illustration.

21. How far should each pupil be required to do the work for himself? Illustrate.

22. Illustrate how you would lead your pupils to take up their arithmetic work as they do the difficulties which arise in their play.

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CHAPTER IV

THE SUBJECT-MATTER OF ARITHMETIC

29. Duality of Subject-Matter. The subject-matter of arithmetic is composed of two more or less distinct elements. The first, the basic element, includes that mass of useful quantitative knowledge which the child needs to acquire for present and future use. When the subject-matter of arithmetic is referred to, this is what is commonly understood and until recently has been receiving nearly all the attention of textbook writers and teachers. The second element comprises all the agencies in the form of games, problems, and so on, through which the basic material is studied.

During the earlier years these two elements are quite distinct; but later the problems partake more and more of the nature of basic material. The teacher who recognizes these two divisions will increase the effectiveness of his instruction because he will distinguish between the essentials and the transitory parts of each lesson, between the real object and the motive used to secure this object.

30. Basic Subject-Matter. The basic matter of arithmetic is to be selected primarily because of its value to the pupils in after life. This must include the language of arithmetic as applied to integers and the various forms of fractions, number relations, and the various principles underlying these; the four fundamental operations; measurements; and the application of all these principles. In addition it should include that vast store of interesting and useful information, dealing with quantitative conditions,

which does not arise in any other grade subject, such as reading of thermometers, reading of architects' plans, postal laws, keeping simple accounts, elements of banking, loaning money, general business procedure, and so on.

31. Elimination and Curtailment in Arithmetic. Due to changes in our custom of living, this part of the subject-matter has undergone continuous modifications since the modern study of arithmetic began. Several topics which were considered important a few decades ago have been wholly eliminated: alligation, duodecimals, Euclidean process of finding the highest common factor, intricate problems on exchange, true discount. Other topics have been greatly curtailed and modified: extraction of cube root; tables of denominate numbers that are seldom used, as troy weight, together with problems requiring many reductions; ratio and proportion (although the superiority of proportion over *unitary analysis,* which has about had its day, is just beginning to be recognized); progressions; partnership.

Material of a general nature such as the following is also being used less and less: problems involving unreasonable numbers, as books weighing sixty pounds, or elephants weighing one hundred fifty pounds; long involved problems, as fractions with uncommon, or large denominators; problems too technical or too mature for the pupils; processes foreign to business practice of today.

Many topics and number conditions are still persisting long after their usefulness has passed simply because of our immobility and fear to break away from the usual custom.

32. The Teacher's Duty Regarding Changes in Arithmetic. Modifications of the material in our arithmetics are much more liable to increase than decrease. It is the teacher's duty to keep pace with such changes as arise in the various textbooks and even to anticipate them with outside problems before these do appear in the texts. A

general proposition which is at present receiving considerable attention is that of bringing the work of arithmetic into closer conformity with business practice. Much of the criticism of arithmetic teaching is due to the fact that this has not been done sufficiently. Such criticism is entirely fitting as there is no reason why the schools should teach arithmetic practices which must later be discarded for others used in actual life.

It often occurs that the practice of the business man is simpler and better than that of the arithmetic, and where such is the case it is the duty of the teachers to adopt the business methods. In case the business men are backward about adopting the most serviceable method, as they are about adopting the metric system of weights and measures, it becomes the duty of the arithmetic teachers to convert them to the better method. It behooves the teacher to be always on the lookout for improvements in his own as well as in the business man's notation and method. As examples may be cited: the business man's computation of the cost of excavating, plastering, and papering. If necessary the teacher should deviate from the text used in order to acquaint the pupils with the present day methods.

33. Reason for Arithmetic Material in Curricula. The main reason for including material in the arithmetic curriculum should be that it will aid the child, at present or in the future, to consider his activities from the quantitative standpoint. Much of the material found in the arithmetic courses at present seems to have been placed there for use in the arithmetic classes only. For instance, the only time that a child makes use of the addition and multiplication tables is in standing up to recite them during class period. In the processes of addition and multiplication a knowledge of the tables is actually a hindrance. This * and many

* See Sec. 118.

similar instances will be discussed later under the consideration of the various topics.

34. A Word of Caution. A word of caution may not be out of place here. Let the teacher always keep before himself that the chief object in arithmetic teaching is to engage the children in arithmetical activity and to accustom them to look at things from the quantitative standpoint. Consequently arithmetic must not degenerate into a training to do a few special forms of computation mechanically. For instance, fractions with denominators other than 2, 4, 5, 8, 10, 20, 25, 32, etc., are uncommon in the industrial and the business world, yet to confine the arithmetical work to these forms would be to give the child a most narrow and superficial idea of fractions, which may be made a vital element in quantitative considerations.

35. Many Should Aid in Selecting Material. The selection of this subject-matter must not be left to haphazard chance, or to prevailing customs in texts. It is the province of authors and teachers of arithmetic and of other educators to get together and select such material as will meet present day needs after they have made a thorough study as to what these needs are. Assistance in making this study can very profitably be asked from men engaged in various pursuits and walks of life.

As each topic will later be considered for detailed study, the various questions of omission, modification, and additions will then receive their proper attention.

36. The Work by Grades. In the main the work of the first four years should be learning the four fundamental operations with integers and fractions, and measures should be taken up to a limited extent. The fifth and sixth years are devoted to a thorough study of measures and of the three forms of fractions, common fractions, decimals, and percentage. Much of mensuration which is now taken late in

the eighth grade may well be studied during these two years in connection with measures. The last two years can thus very largely be given over to the application of the principles previously learned. Excellent material for such applications can be furnished through a study of elementary business procedure, which should form a part of every boy's and girl's education. Commercial usages in computations and short cuts can well be given a systematic treatment at this time.

As far as possible the work of each year should be a unit. This unit is furthermore to encompass the maximum amount of usable material as an aid to the pupil who is compelled to leave school before the eighth grade has been completed.

The following is a résumé of the work given in arithmetic by grades in fifty large cities of the United States as found in Bulletin No. 13, 1911, Bureau of Education, on Mathematics in the Elementary Schools of the United States:

Grade 1. Some nothing or only incidental work—others include addition, subtraction, and multiplication.

Grade 2. Some addition, subtraction, and multiplication—others include division.

Grade 3. Addition and subtraction completed; multiplication and some division; introduction of fractions.

Grade 4. Multiplication and division completed; some fractions.

Grade 5. Fractions; some decimals and measurements.

Grade 6. Decimals; some percentage and interest.

Grade 7. Percentage and applications.

Grade 8. Business applications, proportions, and mensuration; some algebra.

37. Interest. Interest is the prime mover in education. It gives zest to work and imbues the pupils with a determination to succeed. Interest eliminates waste by directing the energies of the mischievous into useful channels. Arousing interest must hence be a part of each teacher's work. But the teacher must not stop here. He must direct the energies

of his interested pupils into useful pursuits. Interest in a school is to be regarded as a means to an end and not as the end itself. For this reason the teacher must be careful not to accept mere fleeting fancy as interest. The boy or girl who needs a continual change of work is not really interested. It is otherwise with the really interested pupil who keeps at a task until he has completed it and objects to any but the most necessary aid.

After all the teacher must be the chief source of interest to the class. Interest in the classroom is very contagious. A bright, alert teacher who is so well acquainted with the arithmetic as to have an interest and enthusiasm in it will have an interested and enthusiastic class. Another fact of which the teacher must never lose sight is that he can cultivate his own interest in arithmetic as well as the interest of his pupils.

38. Motivation. Interest in arithmetic will depend almost wholly upon a proper motivation of the medium in the form of games, problems, and so on, used to teach the number relations and principles of the subject. This has been referred to as the second class of content matter. First of all this motivation must be based upon the law of interest; namely, *that the interest we take in any enterprise varies inversely with the remoteness of the goal from the effort.* The longer we must wait for the results the less interest we take in our labors. The successful teacher will not exhort his pupils to learn their arithmetic because of the future use they will have for it, but will employ the many agencies at his command to give them an immediate object for so doing. It is immaterial what this agency be so long as the interest of the children in the study can be aroused. During the earlier years the appeal to the child may be to learn the number combinations that he or his side may win in number games, that he may play store or run a bank.

As the child grows older and his activities change, the appeal of the teacher must likewise change. During the last years in the grades the children grow into little men and little women. The girls can then easily be interested in the arithmetical side of cooking and simple dressmaking, and the boys in sports and community enterprises.

39. Subject-Matter Should Fit the Season. The important rôle that the changing of the seasons plays in child life must also be recognized in motivating the number work. No interest can be developed during May in problems on Christmas presents or during December on the planting of gardens. Hence it may occasionally be advantageous to change the position of a group of problems in the text. As texts are written to be used during the year, this condition is quite liable to arise in schools having the semi-yearly promotions. A little care on the part of the teacher in modifying the number work to fit the situation will easily set this matter aright.

The selection and use of problems from the other subjects of the grades and outside activities of the children will be discussed under the chapter on Socialization and Correlation of Arithmetic.

IV. EXERCISES

1. Distinguish between the two elements of subject-matter. Illustrate.

2. Point out the two elements in a lesson for one of the first four grades.

3. Consider similarly a lesson for the last two grades. Compare this with that for Ex. 2.

4. Explain the advantages of distinguishing between the two elements of subject-matter. Apply to Exs. 2 and 3.

5. Make a list of what should constitute the basic element.

6. What should predominate in the earlier grades? In the later grades?

7. State the present day tendency in the matter of changing the basic subject-matter together with reasons for the same. Illustrate.

8. What should be the teacher's action in the matter of changes? Illustrate.

9. Explain the suggestions on the selection of the basic material.

10. Discuss the business man's criticism of arithmetic teaching and the teacher's means of overcoming the same.

11. State and explain two business practices differing from the text's treatment of the subject.

12. State and explain the law of interest.

13. State the purpose and value of interest in the schoolroom.

14. How can the spurious be distinguished from the genuine interest?

15. State the teacher's part in securing an interested class. Illustrate.

16. Show the importance of the second arithmetical element.

17. Give six illustrations of motivation for various ages and times of the school year.

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CHAPTER V

SOCIALIZATION AND CORRELATION OF ARITHMETIC

40. Socialization. Our pupils fail to think life in terms of their school work. Our instruction draws upon social phenomena for materials to illustrate the conventionalized subject-matter of the curriculum, but rarely does it use the school subjects as instruments to solve the social problems outside of the school. The so-called applied problems used in the upper grades are in reality not applied problems at all but illustrative problems. They are not introduced for the sake of solving specific and important home and community problems in terms of mathematics, but rather for the sake of illustrating and developing mathematical principles. The problems in the out-of-school life of the pupil should receive primary consideration a part of the time. There is of course a place for the teaching of arithmetic for the sake of preparing for more mathematics, but this should share the time with the socializing of the work.

The ordinary textbook requires the pupil to use separate problems involving the same mathematical principles, but a heterogeneous collection of objects or transactions to be measured. On the other hand, socialized arithmetic would introduce organization into the materials and transactions, analyzing the large problem into many minor ones, while the number combinations are used as the *means*, not the end.

By the very nature of the case a textbook in arithmetic cannot be the all-efficient source of socialized, or truly applied, arithmetic. The text may be very suggestive in

furnishing typical socialized problems but, of course, it never can provide the real problems of any given community.

Arithmetization of the child's life perhaps expresses the desired result more clearly than socialization of arithmetic. The main object is to teach the child to look at his various activities from the quantitative standpoint so that in after life he will think and act in the most rational manner; namely, the quantitative. The thoughtful man and woman consider their acts from the quantitative standpoint; they look at things proportionately; they weigh cause and effect.

41. Illustration. The following problem is intended to illustrate only a single source of socialized arithmetic. This illustration is only an illustration, however, for socialized arithmetic requires, (1) that the problems to be solved are real social problems, (2) that they have a real significance for the pupils, and (3) that the pupils themselves participate in discovering and forming the problems, gathering all data required.

The following problem presupposes a knowledge, first, of what constitutes rent and, second, of the division of expense entailed in improving, repairing, and the general upkeep of the property, also, certain living expenses connected with the property between lessor and lessee. This division of expenses in general is determined by the class of property leased although there is considerable variation in the division of expenses provided for in individual leases. As a rule there are four classes of dwelling places leased and these classes of property carry different contracts as regards the division of expenses. The four classes of dwelling places usually found are:

1. Furnished apartments.
2. Unfurnished apartments.
3. Furnished houses.
4. Unfurnished houses.

42. An Example of the Fourth Class. The following problem deals with an example of the fourth class. To determine whether it was economically a gain or a loss and how much was gained or lost in owning a certain house (located in Seattle, Washington, lots 9 and 10, block 10, University Addition to King County, one mile north of the State University, in a growing district of the city, where property values will undoubtedly increase in the future), from July 1, 1907, to July 1, 1917, rather than renting it for the same length of time, assuming that the house could have been rented.

The data necessary for the solution of the problem follow:

1. The amount paid for the house July 1, 1907, was $3500. In all probability it cannot be sold for more than the purchase price July 1, 1917. Seven per cent is a fair rate for money invested in such property in Seattle. The house would have rented from July 1, 1907, to July 1, 1910, for $30 a month, from July 1, 1910, to July 1, 1917, for $25 a month.

2. The following local improvement assessments were made against the property during the ten-year period. All assessments were due from date of notification and unless paid at the time of notification bore 5% interest. Each assessment had to be paid either in cash or in five equal annual installments plus the interest due each time with the exception of the assessment for condemnation of right of way, which had to be paid at once.

ASSESSMENTS

Oct.	10, 1909—	$125.00	for trunk sewer
May	5, 1910—	65.00	condemnation for right of way to street
Nov.	12, 1912—	110.00	grading street
Jan.	1, 1913—	75.00	constructing sidewalk
May	5, 1913—	100.00	lateral sewer
June	6, 1914—	37.00	for constructing street bridge

IMPROVEMENTS MADE BY THE OWNER

June	5, 1910—	$ 50.00	painting exterior of house
Aug.	5, 1912—	50.00	painting exterior of house
Nov.	1, 1912—	35.00	fencing two sides of lot
Oct.	5, 1913—	125.00	building cement walk
Oct.	20, 1913—	25.00	grading lawn
Oct.	2, 1914—	50.00	decorating interior and varnishing floors
Dec.	2, 1914—	32.00	building playhouse
Sept.	14, 1915—	60.00	painting exterior of house
Yearly average—		40.00	other repairs and improvements

REGULAR RUNNING EXPENSES

Telephone	$2.50 a month
Water averaged	.75 a month
Light averaged	1.50 a month
Gas averaged	1.25 a month
Coal and wood averaged	80.00 a year

43. Sources of Material for Socializing Arithmetic. The alert teacher will find many sources of material for socialized problems. In conformity with the law of interest first comes the quantitative side of plays and games. Foot-races give practice in measurements and computation of velocities. Baseball, basket-ball, and football furnish excellent material for graphs, averages in many forms, cost of equipment, and net costs of games. Commercial activities, such as selling papers, keeping chickens or hares, gardening, working during vacation and hours out of school, and so on, provide much interesting data.

For the later grades there is practically an unlimited supply of community material for problems. In its widest sense here would be included problems referring to the nation, state, and county as well as to the immediate locality. Within this range would come railroads; express companies; postal department, including parcel post and savings deposits; natural resources; manufacturing enterprises; and so on.

A particularly useful source of the more local problems arises in connection with excursions to the various enterprises of the community as stores, bank, postoffice, fire department, bakery, grain elevator, flour mill, creamery, and other more localized activities as lumber mills and so on. A description of each of these excursions furnishes excellent real material for English work. Such a description in order to be complete will require the solution of many problems based upon the data gathered upon each excursion. Many of the sources mentioned above will be considered much more in detail under the various topics where each will naturally fall.

44. Correlation with Other Grade Subjects. Arithmetic is the quantitative side of the grade curriculum. All other subjects as history, geography, physiology, and nature study, together with outside activities as gardening, plays, and games, present a quantitative side which it is necessary to include in a complete treatment of any one of them. Such a full development of the various grade subjects necessitates that they be closely correlated with arithmetic throughout.

Especially is arithmetic to be correlated with present day history and local conditions which are more and more becoming a part of the grade curriculum. So far, practically nothing has been done to take advantage of the exceptional possibilities contained in a rational, common-sense correlation of arithmetic with the other grade subjects. In the place of merely learning facts the children will then be studying these various subjects comparatively. This will give their faculties of observation and discernment constant exercise. Such a correlation will add interest and stimulus as well as educational value to each and every subject of the grade school curriculum while providing the pupils with an opportunity to apply their arithmetical knowledge.

45. An Illustrative Example. One illustration must suffice. Minnesota has been a great lumber producing state. Here arise questions regarding the marketing of lumber such as meaning of board foot, weight of a board foot of various kinds of lumber, number of board feet which can be hauled on a wagon, the number of board feet which can be loaded on a railroad car, expense of shipping, difference of cost of shipment by rail and by water, the number of board feet which could be cut from trees of the size in the school grounds or at the pupils' homes, size of tree required for various sizes of boards and timbers, reason for variation in cost of lumber per M with variation of size, number of board feet which can be obtained per acre, number of board feet required to build the average frame house and hence number which can be built from the lumber off of a quarter section of land, loss in waste by the use of a thick saw in sawing the logs into lumber, cost of keeping up a forest and revenue therefrom 10, 15, 20, etc., years after planting.

Data for most of this material can be obtained from the local lumber dealer, the local railroad agent, and from the Forestry Bureau of the Department of the Interior at Washington, D. C. Problems requiring data which are not obtainable must, of course, be omitted.

46. Value of Such Problems. Problems arising under these conditions will have a flavor of reality which is often lacking in those of the arithmetic texts. For instance, from the data given in the geography that the total output of iron in the United States during 1914 was 30,966,152 T. and for Pennsylvania alone was 9,267,197 T. the pupils can find what part of the total output came from Pennsylvania.

The reciprocal problem, that of finding the total output of iron from the United States for a given period, given the output of Pennsylvania together with what per cent this is of the total, and all similar ones are unnatural as they

never arise in practice. Still many of our arithmetics contain a great number of such distorted problems. The solution of problems selected from other studies and outside activities will do much to remove the stage fright so often shown by pupils when called upon to solve the simplest problem that does not arise in the arithmetic lesson. A proper correlation of arithmetic with the other subjects of the grades will do much to eliminate this sad state of affairs and to place the pupils upon their feet arithmetically.

47. Motivating the Teaching of Accounts in Arithmetic. A judicious reference by the teacher from time to time to the many problems which arise whose solutions are beyond the present arithmetical abilities of the pupils may be used as a stimulus for further study. Such problems can be used very effectively as forerunners of new arithmetic topics. For instance, the class in school gardening may be told that they are to keep accounts of expenses and returns when they are first planning their gardens. This motivates the teaching of accounts in the arithmetic.

Arithmetic will neither be slighted nor given more than its share of time through such a procedure. The day that computation work receives a great deal of time in history or geography the time given to formal arithmetic may be decreased proportionately. In the hands of the progressive, common-sense teacher a socialized and correlated arithmetic can be made a more real, a more vital, and a more usable arithmetic. When it is realized by the teacher that the kind of knowledge the world wants is definite, quantitative knowledge, not vague, qualitative knowledge, this sort of arithmetic will receive more nearly the measure of teaching attention that its importance merits.

48. Sources of Material. The following list is not intended to be at all exhaustive but merely suggestive of the possibilities of problems from the other grade subjects:

Geography

Longitude and time.
Per cent of increase in population.
Value of crops.
Ratios between value of products; one country with another.
Distances between towns; car and freight rates.

Hand Work and Drawing

Measuring material for constructions.
Amount and price of material to make certain articles.

Agriculture

Areas of farms and divisions of sections and so on.
Feeding rations and cost of feed.
Butter fat.
Nitrogen required to raise a 30-bushel corn crop.

History

Finding dates and periods.
Battles; per cent killed and so on.
Financial changes.

Civics

Parcel post.
Support of public enterprises; taxes.

Domestic Science

Per cent of food values.
Changing sizes of recipes, ratio.
Various costs.
Sewing; marking out materials, cost, and economy.

Manual Training

Areas, volumes.
Board measure.
Costs.

V. EXERCISES

1. Explain fully what is meant by the socialization of arithmetic. Compare with the arithmetization of child life.

2. Discuss fully its advantages and disadvantages.

3. Solve the problem stated under 42.

4. Give two other illustrations suitable for grade work.

5. Explain fully how the scheme should be carried out in practice, stating how to obtain the data and so on for any one source mentioned under 43.

6. Explain what is meant by the correlation of arithmetic with the other grade subjects.

7. Develop fully two of the subjects suggested under 48.

8. Suggest at least six other topics. Develop one.

9. Discuss fully how such a socialization and correlation of arithmetic would affect the teaching of it. Is there any danger? What?

10. What may be the danger of giving offence by asking questions of too personal a nature in securing data for problems as suggested under 45? Give illustrations of what would be a proper as well as what would be an improper question.

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CHAPTER VI

PROBLEMS

49. Historical. Problems have always been an important factor in the study of arithmetic but they have not always produced the result that we are asking from them today. During the early years, and indeed, until quite recently, many of the problems found in the textbooks partook largely of the nature of puzzles. They called for a certain shrewdness rather than skill in reasoning. There was, too, little or no attempt at grading or arranging these problems to conform to the ability of the children they were to instruct. They further lacked the elements of motivation and socialization by which we today seek to interest the children.

The commonly accepted distinction between example and problem will be adhered to here and in what follows.

50. Value of Good Problems. Good problems well taught are a vital element of the arithmetic work. First of all, problems are to furnish application of principles and quantitative information pertaining to our industrial and business life. For this reason the teacher must see to it that the use of the principles involved in each problem given for solution is driven home as well as that the problem is solved correctly. Any problem selected from outside of the text must be so presented as to show forcibly the principles which it illustrates.

Problem-solving has another function peculiarly its own; namely, the exercise of judgment it demands in grappling with a given situation. In no other subject in the entire

grade curriculum is the pupil given anything like the drill in judgment that he is given in the solution of arithmetical problems. From the simplest problem, such as that of finding the number of marbles in two boxes, to that of finding the rate of income from bonds bought at a certain price, judgment is necessary in making the solution. The value of such drill in judgment is inestimable. We are daily confronted with situations requiring judgment; in fact, life is one continuous series of problem-solvings. This vital practice in coping with situations which problem-solving affords is one of the great values of problems, especially in the later grades.

51. Solution of Problems. The solution of an arithmetical problem includes four distinct steps:

1. Reading the statement.
2. Deciding what principles to use and how to apply them.
3. Computations and checks.
4. Interpretation of results.

Many teachers fail miserably in problem work because they take the first two steps for their pupils themselves, requiring the pupils to do the computation work only and to check this by looking at the "answer." The fourth step they omit altogether. It may be easier for the teacher to do all this work, but it is not *teaching*, nor is it fair to the children. Really effective work in problem-solving requires, from the earlier grades, application of the four steps mentioned. These will now be discussed in more detail. The teacher who unfortunately has received from a lower grade children unacquainted with this rational mode of procedure may not with fairness "give up the ship," so to speak, and cast the blame upon the former teacher. To the first-class teacher this inefficiency of the former will serve as a stimulus and he will make even more strenuous efforts to overcome the previous teacher's sins

of omission. The real teacher will always be well repaid for such extra effort.

52. Step One; English in Arithmetic. Many a child has been irrevocably lost in his arithmetical work and his parents have thought that he "had no head for mathematics" simply because he did not understand the language used and because this fact was not recognized by his teacher. Often he fails in his arithmetic, not through a lack of arithmetical ability, but through inability to understand the language used—frequently technical—which has never been made clear to him. Most of the difficulties which pupils encounter with stocks and bonds can be traced to their failure to understand the terms used and to the fact that they do not recognize bonds as a form of notes.

To obviate these difficulties, let the teacher first state, or read, the problems to the class. From this the pupils are taught to read the problems intelligently for themselves so as to grasp the meaning contained in the printed page. By this is not meant that the teacher is not to suggest and direct, and even to give illustrations of the proper reading of problems, but that the teacher is not to carry the pupils high and dry over these difficulties of language. The sympathetic teacher will ever be on the alert for difficulties arising from the language used,—either English or arithmetical—and will be ever ready to turn the arithmetic lesson into one of language whenever necessary.

53. Interpreting Silent Reading Highly Important. Silent reading is in much more general use than oral and should, therefore, receive attention commensurate with its importance. One is frequently requested to read an article and to relate the gist of it to others. Excellent practice of reading in this form may be given by distributing cards among some members of the class (see **66**), each bearing a problem. The pupil receiving a card reads it silently; then

he states the contents orally to the class for solution. Thus unconsciously the child is getting practice in the real use of English, that of grasping the meaning from a written page and conveying ideas to others, as well as in arithmetic.

54. Step Two; Exercise of Judgment in Solving Problems. By far the most important step in the process of problem-solving is this one. From the first little problem in the primary grades to the more complicated ones in the last grade the teacher must see to it that each pupil receives drill in exercising his judgment in the selection of proper principles necessary for a satisfactory solution of his problem. Stimulus in the way of questions and suggestions may be helpful but the pupil should seldom, if ever, be told directly how to proceed.

An active exercise of judgment on the part of each pupil means first of all the use of problems well within the mental range of the pupil. Far better for him actually to solve some simple problem by himself, than to make the numerical computations of more difficult problems, after the teacher has given explicit directions as to how to proceed. This is a poor counterfeit in the way of problem-solving. In order to carry out this scheme it may be necessary for the teacher sometimes to modify the problems of the text or to introduce additional simple ones.

55. "Casing Off" Problems. In the older texts each group of problems possessing any common characteristic were placed in a "case" by themselves and the pupils were shown how to solve the problems which came under that one "case." Another group of problems depending upon the same principles were placed in a different section of the text and treated as if they were wholly new and depended upon a special principle for solution. Such highly artificial divisions of the subject-matter destroy the unity of arithmetic and make it appear much more difficult than it is.

The whole structure of arithmetic is based upon a few simple, closely interrelated principles. Each principle appears in innumerable varieties of settings. For instance, finding the cost of a number of like articles; the distance traveled for a number of hours at a uniform speed; the weight of a number of cubic feet of a substance of known specific gravity, and so on, are all based upon the principle of finding the total number of units in a given number of equal-sized groups.

The fundamental point to be observed in the solution of any problem is the principle underlying it and not the particular elements with which it deals. These principles and their uses must be dwelt upon so intensely that the pupils will not look for some trivial earmark by which to identify a problem with one solved on a certain previous page of the text. Only the watchful care of the teacher will prevent the pupils from falling into this most pernicious habit. It is the arithmetical ability and activity which it is desired to develop in the children, not the mechanical memorizing of a lot of rules and descriptions of problems.

56. Inverse Processes. For every arithmetical operation there is an inverse operation. Addition and subtraction are inverse operations; multiplication and division; the changing of a mixed number to an improper fraction and of an improper fraction to a mixed number, and so on. Each operation is thus the reverse, the undoing, of another. Any pair of such operations can be likened to the sewing and ripping of a seam, to the nailing together and the prying apart of two boards, and so on. In the same manner each problem has its inverse. Finding the cost of twelve chairs at eight dollars each is the inverse of finding how many ball games can be attended for three dollars, admission at twenty-five cents. As in the case of operations it is advisable to consider every problem and its inverse simultaneously.

When the class has solved several problems similar to the first one mentioned above the teacher may incidentally suggest its inverse. For example, "How many chairs could then be bought for seventy-two dollars at that price?" The pupils will thus be led to connect naturally, and as a matter of course, the solution of a problem and its reciprocal—the solution of which they understand.

The accompanying scheme is very helpful in bringing this

\$8	\$8	\$?
12	?	12
??	\$96	\$96

inverse relation of problems home to the children. Another inverse relation easily comprehended is that of

$$\text{Length} \times \text{breadth} = \text{area}$$

or,

$$l \times b = a.$$

After the teacher has written down a few such schematic solutions, with either question marks or letters to take the place of the unknown numbers, the pupils are permitted to do this work for themselves. The operation with the given numbers which will give the number to take the place of the question mark or letter will readily suggest itself to the pupils.

These schematic solutions may be carried on throughout the work in arithmetic with excellent success. All such devices are, of course, merely crutches and are to be discarded as soon as they are no longer needed. The average textbook of today gives very little recognition to these inverse relations. Hence much of what will be done will depend largely upon the teacher. Some texts even retain

the division of problems into classes; as in percentage, where three distinct cases are given for the finding of percentage, base, and rate, each of these three cases enjoying a sort of independent principality of its own. Yet here we have exactly the same condition as that given above, three numbers connected by the fact that one is the product of the other two;

$$b \times r = p.$$

57. The Idea of Arithmetical Reciprocity. Recognition of the inverse process will make simple what has heretofore been made complex by the introduction of such artificial machinery as the dividing of problems into classes or groups. These inverse relations are so simple and so natural that they are readily recognized by the children when brought to their notice, and the teacher will find that by directing their attention to some, they will become eager to find others for themselves. This simple idea of *arithmetical reciprocity* has the double advantage of aiding in the rational solution of problems and at the same time it serves to unify the subject-matter.

Furthermore, the number of principles as compared with the number of "cases" is really insignificant. The memory is aided not only by the fact that there are fewer elements to consider but also that these are related. This generalization not only simplifies problem-solving but leads the pupils into a more generalized type of mental activity which is a higher type of mental activity.

58. Step Three; Computation. Pupils should be encouraged to express all operations involved in the solution of problems symbolically before carrying out any computations. Their attention is thus centered at first wholly upon the thinking process and later wholly upon the computations. Cancellation can then be resorted to quite often to

simplify the computations, especially where approximations are acceptable. This is one of the **big aids** in the solution of problems of which advantage should always be taken.

59. Accuracy and Orderliness. Absolute accuracy in calculation should always be insisted upon. Let the teacher accept nothing but 100 per cent accuracy. Solution of problems will be a more effective means of accomplishing this than the use of the usual examples, since fewer problems will be given for solution, and checks can in consequence be required for practically all of them. Illustrations from time to time of the disastrous consequences of numerical inaccuracies will impress the children with the necessity of accuracy in all things. Good checks, it should be remembered, have little or nothing in common with the original computation. As a rule, mistakes are not discovered by a mere repetition or going over of the work again. Preliminary estimates serve as good rough first checks. If, for instance, a first estimate is $25 and the computation brings a result of $253.40, we are almost certain that an error has been made in placing the decimal point. Discussion of checks of the various operations will be taken up herein later in connection with the various topics of arithmetic. Orderliness and system in arranging the work are also conducive to accuracy and should be insisted upon at all times. Problems without answers will be found excellent for teaching self-reliance.

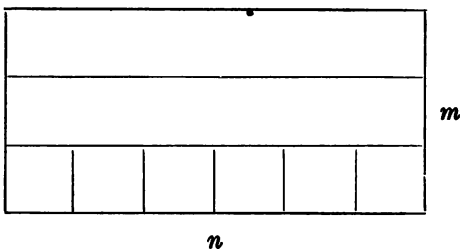
60. Short Cuts. As the work progresses the teacher may give more opportunity in the way of short-cuts for contracting the work. This should not be permitted, however, before the fundamental process, of which it is an abbreviated form, is thoroughly understood. Only by exercising this care can the teacher prevent that artificial knowledge which depends upon the memory, rather than the reasoning power, and is soon forgotten.

61. Step Four; Interpretation of Results. Pupils, and often teachers, consider a problem fully solved when their "answer" tallies with the one in the answer-book. There is little value in such work. Numbers without special designation as to what each one stands for are absolutely meaningless. The speed of a certain train may be designated equally well by 11, 45, or 660, or the cost of a certain book by 4 or 16. In the case of the train, 11 represents feet per second, 45 represents miles per hour, and 660 represents feet per minute. The cost of the book was 4 dollars or 16 marks.

The necessity of such interpretations does not arise in addition or in subtraction, but in multiplication and in division it abounds. The usual process by which to avoid these difficulties is to require that an abstract number always be used as the multiplier. This is, undoubtedly, the surest method for teaching the children in the early number work the necessity of designating final results. In the course of time, however, it becomes quite unreasonable to demand a strict adherence to this rule, as in the case of finding the amount of the daily payroll of a company employing 284 men at $3 per day. No one but the most radical, not to say thoughtless, school teacher would ever think of using 3 as multiplicand.

62. Visualizing and Interpreting. Another means of inculcating in the child the habit of carefully specifying each number is to be found in problems that ask for the areas and volumes of rectangular surfaces and solids. The teacher must again vary the form of solution with the progress of the child. It is quite important that he be made to see, at first, the process used in finding areas and volumes. If the vertical and horizontal lines in the accompanying figure are drawn a unit apart, and if there are n squares in each row and m rows, then there will be $n \times m$ units of

surface in the rectangle. From this we arrive at the conclusion that the *number* of units of surface in a rectangle equals the *number* obtained by multiplying the *number* of units of length by the *number* of units of width. This



general process is then contracted into the formula that "length times width equals area." Volumes are to be treated in the same manner. The fundamental thing to notice is that in all computations in the solution of problems we are concerned wholly in operations with *numbers*. It is always a *number* which is to be found. In the final result this *number* must be properly labeled but it is still a *number* which must be found in the computation. Hence the chief care in the calculatory part of the solution of a problem must be to perform the operations in the best possible manner and never with the use of such distorted operations as the one exemplified in the accompanying illustration.

$3
285

The value of proportion as a numerical operation often suffers by too rigid enforcement of so-called "rational" definition of ratio. It is not loosening up in this phase of problem-solving that is advocated, but rather a broader and a much more common-sense treatment of it.

63. Formulation of Problems. Successful teachers give more problem work than is found in the average text. Such

problems will be found very helpful while their formulation will not be nearly so difficult or so irksome as might be expected.

64. Changing Numbers in Textual Problems. Simply changing the numbers in the text furnishes the resourceful teacher with great possibilities.

It means first, an unlimited number of problems at his disposal which can be made simple, or difficult, at will.

Second, board work can thus be made to give practice in new number relations and yet not be a repetition of the seat work.

Third, much excellent drill is afforded by substituting numbers which come within the mental range of the pupils. This will result, not only in mental drill, but in a greatly increased number of applications of the principles involved. For instance, after one pupil has explained from a board-solution how to find the interest upon $560 at 7% for 5 months, the teacher may ask the class to find, mentally, the interest upon $500 at 8% for 6 months, for 3 months, and so on.

Fourth, the dependence of the children upon answers can be removed by this kind of work.

Fifth, by changing the numbers of the text and making the assignment on the day of the recitation the teacher's greatest handicap to good arithmetic teaching is removed; namely, the prevalent injudicious, incompetent home "help." By far the greatest hindrance to good teaching of arithmetic is this continual home "help." It furnishes the basis of much criticism upon our arithmetic teaching, and that, too, by the ones who do most to furnish its cause. Such a treatment of the work in the text as suggested above will remove the great hindrance of home "help," and for that reason alone will be found to be very much worth while.

65. Office of Difficult Problems. In the formulation of problems the teacher must recognize that problems are like puzzles; if too simple they do not hold the attention, and if too difficult they discourage all ambition to secure a solution. The danger of selecting problems that are too simple, however, is not nearly so great as that of selecting those that are too difficult. It is far better to have every member of a class solve several simple problems than to have them merely attempt a few difficult ones. Solutions of problems based upon simple everyday data are the best for driving home the arithmetical principles. A few of the more difficult problems wisely interspersed serve as a stimulus to the brighter pupils. These can be made optional, thus giving practice for those pupils who are brighter and more ambitious than the average of the class. It is seldom that a class even if it contains only five or six pupils is so well balanced that the same number of problems is an equal assignment for all. It is the duty of the teacher to adapt the work to the needs of the bright, alert pupil as well as to the slower ones.

66. "Fake" Practical Problems. The formulation of these outside problems needs in no way to become a burden to the teacher. The ability to change the numbers and to dictate oral problems similar to those in the text will come with a little practice. If desired the numbers substituted for those in the text may be written in the margin.

Data for local problems can and should be collected largely by the pupils under the direction of the teacher. Data from elsewhere can be secured through letters written by the pupils. This data is then written into the form of problems upon cards which the teacher preserves and uses from year to year. Additions made to this card system keep it growing and up-to-date. A master-card containing the computed results may be included in the file. The copying

of these cards for the teacher's file can be turned into "real" writing lessons.

It is, of course, unreasonable to expect that all problems be based upon actual data but all problems should be based upon conditions that are reasonable. Such data as the following are not reasonable and therefore should not be used: a farm wagon weighing 150 pounds, street paving costing 10 cents per square yard, a baseball "diamond" 110 feet square, and so on. These are all illustrations of "fake" practical problems which are to be guarded against at all times.

67. Unmotivated Problems Are Impractical. The teacher must beware of many of the so called "practical problems." Many problems which masquerade under that name are only poor imitations of what the child is supposed to run across in the future. They miss, altogether, the first essential of practicability; namely, motivation. Problems dealing with the present advanced cost of commodities over their cost fifteen years ago may furnish excellent practice in subtraction, but such problems are highly impractical to a class in the third grade because they do not appeal to the child's interest.

All good problems must be practical in the large sense of that word; in addition to giving practice in number combinations, principles, and the various operations, they must come within the pupils' comprehension, in order to give them opportunity to exercise their judgment and they must be properly motivated to hold the pupils' undivided attention.

VI. EXERCISES

1. Compare the problems found in the texts in the past with what is sought through problems today.
2. Distinguish between problems and examples.

3. Explain and illustrate the use of problems in the teaching of arithmetical principles and operations.

4. Explain and illustrate the need of judgment in the solution of problems.

5. Explain the value of problems in developing judgment.

6. Solve a problem to illustrate the four steps in problem-solving.

7. Why do so many teachers fail in teaching problem-solving? Illustrate.

8. What should be the procedure of a teacher in, say, the sixth grade whose pupils have had no adequate preparation in problem-solving? Illustrate.

9. What phase of English enters into problem-solving? Discuss fully and illustrate.

10. How would you proceed to cultivate this power in your pupils? Illustrate.

Read and explain the conditions given and state the results desired in the following nine problems:

11. A lawyer remits to his client \$608 which is the face value of a note he has collected less 5% for collection. What was the face of the note?

12. A man spent 15 cents less than $\frac{1}{3}$ of his money and had \$3.75 left. How much had he at first?

13. What number divided by $\frac{2}{3}$ of 60 will give $\frac{7}{8}$?

14. Corn was sold at 65 cents per bushel which was a gain of 10%. How many bushels were sold if the total gain was \$208.81?

15. How much school tax will a man pay who owns \$24,500 worth of property if the school tax is $\frac{2}{5}$ mill?

16. In an election requiring 60% of all the votes cast to carry a bond issue 120 voted "yes" and 40 voted "no." Did the election carry by 24 or by 60 votes?

17. How much water must be added to 100 cc. of 90% alcohol to make it 60% pure?

18. A house heated by a hot air furnace has two heating pipes with 10" and three with 8" diameter. If the area of the cross-section of the return pipe is to be $\frac{5}{8}$ that of the total areas of the hot air pipes, what will be the diameter of the return pipe?

19. A sailor is climbing a mast at the rate of 6' per second. Find his actual rate of motion if the ship is sailing at the rate of 12 miles per hour.

20. Suggest two other problems for reading and interpretation.

21. Discuss fully the degree of difficulty to be permitted to enter into problems.

22. Why should problems not be classified into "cases"? Illustrate.

23. Where should the emphasis be placed in teaching problem-solving? How can this be accomplished?

24. Suggest at least three quite dissimilar problems depending upon the same principle for solution.

25. Explain fully the nature of "inverse processes." Illustrate.

26. State the inverse problems for 11-19 given above.

27. Discuss the advantages of the inverse processes in problem-solving. Illustrate.

28. Explain fully how you would introduce the inverse processes to a class.

29. Why should all computations be indicated before any of them are carried out? Illustrate.

30. Apply to problems 13, 14, 17, 18, and 19 above.

31. In grading problems what importance should be given to accuracy? Illustrate the disastrous consequences of inaccurate computations.

32. What are good checks? Illustrate.

33. Explain the value of making preliminary estimates before carrying out any computations. Illustrate.

34. Make preliminary estimates of results for problems 11-19 above.

35. Explain and illustrate how computation work is to be increased from year to year.

36. Discuss fully the necessity of interpreting final results.

37. Interpret the results of solution of problems 11-19 above.

38. Explain how to treat this fundamental phase of problem-solving.

39. Explain the changing of the numbers in the problems of the text and its use. Give three illustrations.

40. Discuss the handicap resulting from "home help." How can it be overcome?

41. What are the primary essentials of good problems? Illustrate.

42. Criticize the loose use of the term "practical problems."

43. Discuss and illustrate the real meaning of practical problems for the various grades.

44. Discuss the method of gathering data for outside problems and of supplying such as is not obtainable.

45. Formulate a problem based upon such data.

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CHAPTER VII

DRILLS

68. Necessity for Reviews and Drills. It is a good plan to begin each year's work with a quick review of the previous year's work, especially of those parts which will arise during the coming year. Short reviews may come at the end of each topic, but the principal one generally comes just before the close of the year. This should serve to unify the matter studied during the year as well as to fix it well in mind.

Good arithmetic work requires daily reviews through the various forms of drill. The successful teacher of arithmetic makes constant use of drills. In no other way can so much work be accomplished in the same number of minutes. With the full schedule of today this is a large item. Number facts and their application in simple operations must be habituated so thoroughly in the child's mind that he uses them as spontaneously as he does the letters of the alphabet to form words. Ceaseless repetition through intelligent use of daily drills is the only way such habituation can be attained. Much can also be done in these drills to connect the various parts of the work into a unified whole, which is another **big thing**; an essential in successful teaching.

69. Mental Calculation. A large percentage of the arithmetical computations that the average man or woman is called upon to perform are so simple that they can easily be carried out mentally. As a rule, however, children show

utter bewilderment if called upon to do the simplest computation mentally, even though they can carry out the most complicated operations upon paper with the text before them. This inefficiency is justly criticized most severely by the business men of today. It behooves the teachers to remove the cause through the use of much oral work, including approximations. There is no desire to produce mental acrobats by means of oral drills in the solution of such problems as the following, which were prevalent in the mental arithmetics of a few years ago:

1. A and B can mow a field in 10 days and A can do $\frac{2}{3}$ as much as B. What part of the field will each mow in one day? How long will it take each to mow the field?

2. At what time between six and seven o'clock will the hands of a clock be at right angles to each other?

However good the teaching of arithmetic may otherwise be, and however much time may be devoted to it, if the drill work is neglected the arithmetical development of the child will fall far short of his present-day needs. The time and energy expended in drill work will be found an excellent investment in the general mental development of the children, and the development of this mental efficiency will do as much as any one thing to turn the modern business man from an adverse critic to an ardent supporter of our public schools. The teacher who would succeed in the teaching of arithmetic, therefore, is urged to study the questions relating to drills and later to apply these drills in the schoolroom with energy commensurate with their possibilities.

70. Essentials of Good Drills. It is not drills but *good* drills which will produce the desired results. They must first of all be quick, snappy, and full of life, such as will keep the pupils constantly on the alert. The life and enthusiasm exhibited by the children will be directly propor-

tional to that of the teacher. A slow, listless teacher who must hunt for the next drill question will have a class of slow, listless pupils who require much time to hunt for answers to his questions, or the class may become inattentive and restless, and so cause the teacher worry and annoyance. Sluggish drills are barely better than no drills. Oral drills must require quick, spontaneous answers. In written drills the dictation should be either so rapid that none of the pupils can give results before all are ready to work, as in addition examples, or should be slow enough to permit the result to be given after each example, as in the multiplication of integers.

71. Drill Exercises Must Engage Activity of All. Not only must the drills be snappy, but they must be such that every child is engaged in number activity every minute of the time. It is well too for the teacher to focus the attention upon such number facts and operations as have lately been studied and upon those which the children find the most difficult. To do this the teacher must be on the daily lookout for difficulties which the children still have with what they have previously studied. On the other hand, the easier operations and number facts must not be wholly neglected, but must be mixed judiciously with the more troublesome ones. The child is thus given a review of all the processes to be habituated while the variety serves to hold his active attention throughout and to create animation during the class period.

72. Drills Unify Subject-Matter. Drills can also be made very useful as a means of unifying the subject-matter. The thinking teacher may do much to connect all parts of the work. New work can be introduced by drills and oral problems on the old with which it is allied. Those number forms and principles which often arise together in practice should be joined together in drills.

73. Drill Periods Must Be Brief. Vigorous, snappy drills will be quite fatiguing and the accuracy of the children will soon fall off perceptibly. Hence, drills must be continued for only a short time and never after the children begin to give unreliable results. Ten minutes is sufficient for drills in the upper grades and the time must be shortened proportionately for the lower grades.

74. Forms for Drills. Space will permit only a few illustrations of forms for drill. Additional ones and modifications of these to suit local conditions can easily be worked out by the teacher. First comes dictation by the teacher. When the pupils are working at the board it is well for the teacher to stand near the middle of the room. At other times he may be seated or may stand before the class. In examples involving much labor, as long additions, and in seat work requiring some time, the teacher must give numbers from a text or paper, in order to be able to read the correct results later.

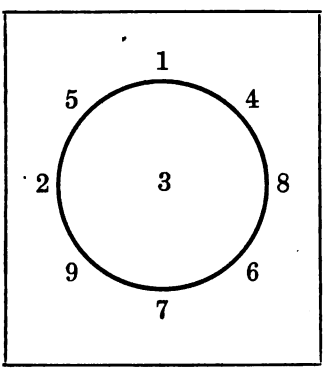
Printed or mimeographed number forms for drills save the teacher and greatly increase the efficiency of the work. Dictations without reference to book or paper are strongly advised for other times. The teacher is thus able to give his undivided attention to the class and will gain the respect of all the pupils for having this command of the subject-matter. Let it be borne in mind that this does not mean aimless drills in which the work is chosen hit and miss on the spur of the moment. Neither does it mean long, laborious preparations, but a definite idea as to what is to be accomplished in the time allotted for drill.

The inexperienced teacher may at first jot down a skeleton outline of some drills upon a card but this will soon become unnecessary. In giving drills from outside the text the teacher must be careful to introduce nothing beyond what has been studied unless it be to introduce a new fact.

75. An Example. A very simple drill is for the teacher standing before the class to say, without reference to book or paper, "4, add 6, divide by 2, subtract 3," and so on, and then ask for the result. The number operations vary, of course, with the grade, and the speed can be increased with practice,

76. Using Drill Cards. "Flash cards" containing some number fact, as 3×5 or $4 + 7$, printed upon bristol board with figures sufficiently large to be seen distinctly from any part of the room can be made the basis of excellent drills. As the teacher holds up a card for momentary view by the class any child can be called upon to give the result. Hands can be called for, or the children may each write the results upon papers which are later exchanged and corrected.

77. Skillful Arrangements of Numbers for Drill. In many of the best drills, the numbers are in constant view.



This is as it should be, as the numbers are usually seen in most mental operations. Such are the following two forms:

Numbers are placed in any order on a circle with a number in the center. As the teacher points to a number on

the circle, the children give the sum, product, or the result of whatever operation is called for, of this number and the one at the center.

In another form, numbers are written on the left of a vertical line. The two at the top are added and the sum placed on the right side of the line opposite the second. The third number is then added to this sum, and so on down the line. The last sum will, of course, be the sum of the first column. If one number is repeated 5 or 10 times the teacher can give the final result at a glance. One child may be sent to the board to write in the results given by the children or each child may write down the column as the teacher dictates.

15	
3	18
6	24
9	33
12	45
35	80
7	87
8	95
?	?
?	?

78. Other Devices for Drill. Problems and examples placed upon cards which are given out to the pupils to work can be used as a sort of drill. A record is kept of the work done by each pupil and he is given extra credit for the same.

A most useful exercise is that of giving problems similar to the written work but using numbers which can be handled mentally as was previously mentioned in **64**. Such problems require little time, give much practice in the principles involved as well as in numerical work, and hold the class alert at all times. This is one of the teacher's best means for producing good arithmeticians and one which should be used continually. The composition, off-hand, of such problems may seem a difficult matter to the inexperienced teacher, but will be found quite simple after a little practice. In order to gain self-reliance, the new teacher may at first prepare a set of such problems before class time and gradually learn to dictate them as the lessons proceed.

79. Number Games. Number games provide excellent forms of drill, especially in the early grades. A few *good* games will be found to be much more useful than a great variety. Children do not tire of repetition and many forms are likely to become confusing. Every good number game must, (1) be easy for the teacher to operate, (2) give life to the class, (3) create a wholesome rivalry among the pupils, and (4) include all of the pupils in the number activity throughout the entire game.

80. Group Competition and Monthly Records. Often a game which is very successful in a large class will be almost useless in a small class, and *vice versa*. Games which include only a part of the children or in which a child drops out as soon as he fails to answer, as in the old "spell-down," misses one of its chief purposes, that is, of *affording drill to those needing it the most*. Wholesome rivalry may be created by dividing the room into permanent teams for group competition and by keeping records of each child for a certain time, such as a month. The outside coaching thus received by the poor pupils is often a more effective stimulus than any other the teacher can provide.

81. Keeping Individual and Team Scores in Drill. The flash cards just mentioned can be used with excellent results either for individual or team records. In case the children work on paper each child is given one point for every correct result to be added either to his own individual score or the team score.

In a room with more than one grade all may play at once, providing only such number facts are used as each child has learned. This will be an added incentive for the children to learn the combinations known by any of the other pupils in the room. The younger children in a rural school can be supervised by one of the older pupils in such games, thus accomplishing a twofold purpose.

82. A Team Game for a Small Class. The following is an excellent team game for a small class: The teacher writes upon the board as many columns of number combinations or other questions as there are teams playing. Each column contains the same number questions but in a different order. Each team then forms in a line in front of a column of questions. The captain goes to the board and writes the answer after the first question. He then gives his piece of chalk to the pupil next in line and takes his place at the end of the line. The process is repeated by this second pupil and so on through until every question in the column has been answered. In order to interest all of the pupils in every operation each pupil is permitted to correct any error made by one of his team mates. If a pupil is unable to answer his question he writes his name upon the board.

At the conclusion of the game the teams are scored as follows: The team finishing last is given 10, the one next to the last 20, and so on, from which is subtracted 1 for each error remaining and 2 for the name of each pupil unable to answer the question which fell to him.

The daily record may be kept for a month at some place upon the board, or on a suitable paper. For a rural school it may be modified by placing the children of lower grades together, then those next in ability, and so on, to compose the various teams. The same questions are not given to all of the teams but such as each team can do. The time and score is otherwise taken in the same manner.

Such games as these have a threefold advantage; they afford drill in number work, they possess the stimulating element of keen motivation, and they serve to promote the social spirit of the classroom, that spirit of fellowship in coöperation to which every child responds. In the present stage of our civilization this spirit needs nurturing.

83. A "Field Meet." A very useful game is one that may be referred to as a "field meet." The room is divided into teams if that has not already been done. Exercises are dictated by the teacher and copied by the pupils if printed or mimeographed copies are not available. Work begins and stops at a given signal from the teacher. Papers are exchanged and corrected as the teacher reads the correct results. Each correct result is given one point towards the total score. The "meet" is continued for several days, one event, as addition, being "run" each day. This form of game is especially useful at the beginning or end of the school year when a general review drill is needed.

Card games based upon the four fundamental operations with integers and fractions can be procured from educational supply companies.

84. Approximations. Approximations are desired fully as often as exact results; yet very little attention is given to this useful part of arithmetic. It is exceptional to find a boy or girl who can give the approximate cost of 17 pounds of meat at 23 cents per pound, or who can tell about how many pounds can be bought for $2 at 35 cents per pound. Such problems will give practice not only in arithmetical computations, but also in discrimination and self-reliance. The value of a continual use of such problems can hardly be estimated. Approximations also afford a rough check on written work. For instance, the circumference of a circle of radius 13 is the product of 26 and 3.1416. This is more than 25×3.1416 but quite near 25×3.2 which is $\frac{100}{4} \times 3.2$, or 80. The most successful practice in this very useful part of arithmetic is obtained by requiring approximations for results of most problems to be solved, before the actual work of solution, as was mentioned in **59**.

85. Value of Tests. Short tests coming frequently are most beneficial to both teacher and pupils. Those which

come unannounced are generally most productive of results. The benefits to the teacher may be summarized as follows:

1. Tests give written evidence regarding the pupils' daily preparation.
2. Tests tell which pupils are lacking in self-reliance.
3. Tests show what work needs further emphasis.

As should be the case, the pupils are the greatest benefactors. These again can only be mentioned:

1. Tests prove an incentive for daily work to the less industrious. This the teacher can enhance by including one or two problems from the day's lesson.
2. Tests bring home to the pupils their weaknesses.
3. Tests show the pupils their weaknesses when thrown upon their own resources.
4. Tests develop self-reliance.

VII. EXERCISES

1. What should be the object of reviews in arithmetic?
2. Discuss fully the review to be given at the beginning and at the end of the year.
3. Discuss the business man's criticism of arithmetic teaching of today. Is it justifiable? Illustrate.
4. What can be done to overcome this criticism? Illustrate.
5. Compare the mental arithmetic of a few years ago with the demands of today.
6. Discuss the value of daily drills.
7. What should be the material for daily drills?
8. State and discuss the essentials of good drills. Illustrate.
9. When and for how long should drills be given?
10. Discuss and criticize the drills outlined in 75-78.
11. Suggest two other forms.
12. Suggest how the latter or those mentioned in 75-78 can be modified to satisfy some given school condition.
13. Give an illustration in which paper or book need not be used on drills. Make out a card as suggested.
14. Explain the use of short drill problems similar to those with longer numbers solved upon the board. Illustrate.

15. State and discuss the essentials of good number games.
16. Discuss the games mentioned in 81-83.
17. Suggest two additional games.
18. Suggest modifications for the latter or those stated herein to satisfy some given school condition.
19. What is the value of approximations? Illustrate.
20. Give the approximate results of the following:
    i. Find the cost of 19 yards of cloth at 14 cents per yard.
    ii. If sound travels 1120 feet per second, how long will it take it to go 3 miles?
    iii. What will 235 bushels of corn cost at 75 cents per bushel?
    iv. What is the interest on $635 for 5 months at 6%?
21. Give three problems suitable for approximations in the grades.
22. What is the greatest weakness of teachers in conducting drill work? Illustrate.
23. What is the object of tests?
24. State the essentials of a good test.
25. Prepare a test on any topic of grade arithmetic.

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CHAPTER VIII

MISCELLANEOUS

86. Foreword. The present chapter will contain some questions, together with a few general suggestions, which for various reasons have not been included under the previous discussions.

87. Mode of Thought and Language of Arithmetic. Arithmetic possesses a mode of thought peculiar to itself by which it answers quantitative questions of relation such as "how many?" or "how much?" It has also its own language, which is as distinct a language as is English, French, or ancient Greek. The process of thought in the mental development of the race is represented by three stages: (1) the thought, (2) the statement of the thought in terms of symbols, and (3) the use of this symbolical language as a means of deriving further ideas and of showing the relations between ideas. Counting and simple arithmetical operations had been in use for centuries before the Hindus invented the present number-system.

88. Language and Thought Must Be Distinguished in Teaching. Successful teaching of arithmetic requires that the teacher recognize, consciously or unconsciously, this difference between the mode of thought and the language of arithmetic. For instance, common fractions, decimal fractions, and percentage are all fractions; $1\frac{1}{5}$, 1.2, and 120% are identical statements. Decimals and percentage are new notations expressing old ideas. Many of the difficulties encountered by children in their study of arithmetic

are owing to the fact that pupils are required to learn the thought and the language simultaneously. This attempt to do two things at the same time leads to confusion in which the notation is often taken for the main object; the symbol for the idea. The pupils will be prone to consider the symbolism the real thing, and the idea, a hazy mystery. They will, therefore, direct their attention towards memorizing the phraseology used by the teacher and the book, together with the mechanical manipulations, in much the same manner that one unacquainted with the Chinese language would memorize a declamation or the writing of a letter from a copy in that language. The pupils thus imitate in a parrot-like fashion certain symbols which they mechanically use with no comprehension of the principles involved. Consequently they are unable to apply these principles under slightly different conditions, and do their arithmetic work in a sort of unreasoning frenzy, their aim being merely "to get the answer," and finally they give up in despair, because "they have no head for mathematics."

89. Confusion of Children with Fractions. Much of this mental confusion on the part of the pupil may be avoided by rational teaching in which the teacher takes into account the above mentioned process in the mental development of the race. The pupils, then, are not bewildered by having to do two things at the same time. A few years ago fractions were introduced by means of the text alone without any objective work. Various fractions were taken up in their symbolical forms as $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{16}$, etc. In a short time the pupils were given a mechanical scheme for changing several quite varied fractions to others having a common denominator. Yet a little later they were told to add fractions by changing them to fractions with a common denominator and then to add the numerators. The child who became mixed had "to scheme around" until he "caught on" as to

how to proceed. Some pupils would add $\frac{2}{3} + \frac{4}{5}$ by placing the sum of the product of each numerator and the denominator of the other fraction over the product of the two denominators. When these pupils met the explanation in the text of $\frac{2}{3} + \frac{4}{6}$ they were confused and when they were required to add three or more fractions they had to learn an entirely new process.

The progressive teacher of today first places emphasis upon the *idea* of fractions, after which the meaning of the symbolism is taken up with equal thoroughness. From this the child is led to see that one-half and three-sixths have identical meanings as well as that these are expressed symbolically by $\frac{1}{2} = \frac{3}{6}$. For further discussion of the progressive method of teaching fractions, see **147-151.**

90. Symbols of Arithmetic. As the necessity for discrimination in the choice of English, both in stating problems and in interpreting final results has been discussed, space will not be taken here to recapitulate. The number system has also been treated as fully in Chapter II as a work of this nature will permit. The requisites for a good system of symbols are *consistency, clearness, simplicity,* and *universal use.*

91. Some Cautions to the Teacher. It is best, at all times, to avoid such forms as are inconsistent with common usage as that found in the accompanying diagram illustrating the addition of fractions. Denominators should never be placed above the numerator in any common expression of fractions, as has here been done. Too many forms are generally a detriment because they are confusing. One good form for each idea and operation, together with insistence by the teacher on its use, will be found best.

		12
$4\frac{1}{3}$	$= 4$	4
$2\frac{1}{4}$	$= 2$	3
$3\frac{1}{6}$	$= 3$	2
	9	$\frac{9}{12}$

New symbols and their use must be learned just as carefully as new principles. Incorrect, slipshod statements should never be accepted by the teacher thereafter, even though they "can be understood." As an illustration, may be mentioned:

$$3 + 5 = 8 \times 7 = 56 \div 4 = 14.$$

The pupil should master the symbols and forms of operation so thoroughly that they give him no conscious attention in carrying out any operation. In this connection, it is well for the teacher always to insist upon good figures and general neatness. As a prerequisite, the teacher, himself, must set a good example in this matter. Systematic neatness is the first requisite of accuracy, and accuracy is the very foundation of computation. A further value is the habit it inculcates in the child, of being generally systematic and careful in doing anything, particularly in that of making a permanent record,—a habit which will be invaluable to him throughout his life.

92. Unification. Basic arithmetic is taught largely as a series of wholly unrelated topics. It is, however, founded upon comparatively few principles and operations, together with their symbols, and should for this reason be taught as a unified whole. A unification of the subject-matter in arithmetic is as necessary and as much to be desired as unification in the teaching of History or English Literature or Geometry. It will mean increased clearness, greater thoroughness, and, in the end, a great saving of time. Until recently, this phase of the teaching of elementary arithmetic has been given insufficient attention; but the great possibilities resulting from such a method make it one to be urged upon the teacher who has not only the welfare of his pupils at heart, but also his own selfish interests. It is safe

to say that no other method of teaching will be productive of results so real and so lasting.

93. Early Instances of Unification. Number relations represent one instance in the unification of arithmetic which is most serviceable, especially in business computations. The successful *use* by the teacher, of number relations will make his pupils keen and alert as well as accurate in computations; and thus secure for them and for himself the commendation of the business men of his community. As these number relations can be treated better in connection with the discussion of the fundamental operations with integers and fractions all that is necessary here is to call attention to their value by a few simple illustrations. In the first place, it must be borne in mind that every number fact and its related facts are to be studied together. Thus with 7×8 should come 8×7, $56 \div 7$, $56 \div 8$, $\frac{1}{8}$ of 56, and $\frac{1}{7}$ of 56. The pupils are, in this way, required to learn only one basic fact, $7 \times 8 = 56$. The other number facts, or relations, will be easily grasped from their connection with this primary one, and grouped around it. Again, adding 9 to any number amounts to increasing the tens by one, then deducting one unit. Hence, aliquot parts together with all short cuts, such processes as multiplication by 11, 22, 25, $9\frac{3}{4}$, $10\frac{1}{4}$, $6\frac{1}{4}$, 484, and so on, are based upon a unification of number relations.

So much for the practical value, or rather, the necessity, from a practical point of view, of unifying the teaching of arithmetic in all its phases. Let it be borne in mind, however, that spasmodic drill in this method of grouping or unifying number relations will bring little or no result. The teacher must be constantly on the lookout for them as they recur. And if the textbook does not supply sufficient material for such practice,—as it doubtless will not,—the teacher can easily supply additional problems, or simply

change the numbers in the problems of the text. In any event, drill in these number relations as family groups should be continued with unrelaxed zeal day after day. This, however, does not refer to the countless short cuts which are considered in the business arithmetic suggested for the upper grades. This practice of unifying the various phases of arithmetic may be carried further; for example, every operation, with its inverse, as subtraction with addition, division with multiplication, and evolution with involution. It is much more natural, and, hence, easier for the child to learn such operations as subtraction and division when they are presented to him as the inverse of operations with which he is already familiar. Similarly, the good old rules for placing the decimal point in the multiplication and division of decimals are inverse operations and should be so taught.

94. Later Instances of Unification. Indeed the meaning of any operation will be grasped much more readily and much more clearly, if it is connected with one already understood. Application of the idea of inverse operations to problems involving such fundamental relations as the percentage equation, $r \times b = p$, will do wonders in simplifying the solution of such problems. (See **198.**) The use of recurring principles was also treated amply in **55** and will not be repeated here.

95. Unification a Matter of Teaching, Not Textbook Writing. From the foregoing discussion it is evident that for carrying out this plan successfully no special texts are necessary but specially prepared teachers are needed. These teachers must be sufficiently alert to recognize recurring ideas and principles together with the relations existing between them and also be resourceful enough to bring the mutual relations home graphically to the children. The following are some very brief suggestions to the teachers

for increasing their efficiency in the teaching of arithmetic through unifying the subject-matter:

1. The first requisite is a comprehension of the language. This includes the ability to use the symbolical arithmetical language as well as the ability known as a good command of English. New terms should be tactfully connected with the old until the meanings of the new ones are grasped; after which these may be used so frequently as to become a part of the pupils' own vocabulary.

2. Stress should be laid on principles, rather than classes or problems, and the pupils taught to be on the constant lookout for recurring principles and number relations as they occur in new problems. If reciprocity, i. e., the inverses of operations, be introduced early in the work, it will only be a short time until each operation will suggest its inverse in the mind of the pupil.

Constant watchfulness and the stimulus supplied by a keen teacher will work wonders in developing his ability for seeing these number relations and thus in the end will simplify the general work and save time. Problems without numbers in which the operation only is stated afford excellent practice in this connection, because in these the pupil's attention is concentrated upon the process rather than upon the numbers and "the answer."

3. New topics should *dovetail,* as it were, with those already understood. Thus the pupil is made to glide into new topics incidentally without experiencing the shock of newness. While such introductions are incidental they must be planned with careful foresight by the teacher, and be far from merely accidental. They should come along as *episodes* in the main current of thought.

4. Constant reviews are a necessity. These can be made to count the most, by taking stock from time to time of the class work and so ascertaining what has been neglected and

what needs particular attention. A word is timely in this connection, concerning the abandoning of one topic with the taking up of a new one. This deplorable tendency doubtless arises from the perverted idea of the teacher that such a course is necessary to cover the required work within the time limit. The use of inverse processes and unification of the work, as tried out in the training school of a well-known normal school, actually cut down the time required to complete certain prescribed work instead of lengthening it.

5. All teaching should be for future use and application. With this end in view, every principle and process which will be applied in the work of succeeding years should be given special attention. To teach thus, with one eye on the future requires, of course, a thorough knowledge of the work of the succeeding years as well as complete sympathy with it. At the same time, the arithmetic teaching of a school can be coördinated into one unified whole, each teacher working for the arithmetical development of the child for all the future as well as for the present grade.

96. Lesson Plan. Each lesson should be a unit in itself and have its own object. It is well to hold to this rule quite strictly throughout; not so strictly, however, as to make it impossible to consider a matter that may arise during the recitation in which the pupils are really interested, but which the teacher had not anticipated. Rules should never be so iron-clad as not to admit of some flexibility when circumstances demand it. At the same time, the work ought to be amalgamated or, as said before, dovetailed with that of the previous lessons as far as possible. Unification is the thing to strive for. To this end, let the teacher give some consideration to each lesson before taking it up with the class. Whether this be burdensome or not will depend largely upon how much use the teacher has made of the page margins as previously suggested in **66.**

Preparation need not mean the solving of all problems in the lesson assigned but it must mean becoming acquainted with the work in general and laying plans for action. It resolves itself into the twentieth century slogan of "preparedness."

97. Lesson Assignments. The matter of lesson assignment demands consideration commensurate with its importance. This it rarely receives. In no subject do the lesson assignments need more attention than in arithmetic where each lesson depends so much upon the preceding ones. Assignments should never be made hurriedly. It is suggested that they be made during the course of the recitation and be connected with it as far as possible. Review work may be brought up, if necessary, in order to find something by which to connect the next lesson. New work is thus given a purpose, or direction; it is, at the same time, shorn of some of its strangeness, and all the work is brought together into a unified whole. A wholly new topic may thus be introduced incidentally with as little as possible said about its being new. Much is not really new—there is nothing newer in counting by quarter inches than in counting by whole inches. Careful preparation on the part of the teacher is, however, necessary in order to make assignments and introduce new material thus discretely and casually.

98. Waste of Time. Too often teachers waste time that, if put to proper use, would transform their pupils from very mediocre to proficient arithmeticians. First, there is the waste at the beginning of the recitation in giving out papers which can largely be avoided. There are the needless repetitions, as in requiring the children to say, "Three times five equals fifteen," instead of merely "fifteen."

During drill work at the board, some teachers assign further examples only when every pupil has completed

those already assigned. Time is also lost by permitting pupils to go to their seats after they have completed a problem at the board and allowing them to sit unoccupied until all the pupils have completed their work. The teacher may also be assisting a single pupil while the others are waiting for additional work to do. Listlessness, and a lack of interest will be the result in each of these cases. There should be *no cessation of work* at any time during the arithmetic lesson. "Explanations," in which a pupil points to the numbers upon the board as he reads are generally a pure waste of time. This, however, does not apply to the pupil who enters into the spirit of the class and explains energetically just what he has done and what is his result. Lengthy computations affording little practice along needed lines are a waste of time unless they provide a much needed review. Much time is also lost by the pupils attempting to learn lessons involving processes and language with which they are almost wholly unfamiliar.

99. How to Save Time. In the rural schools much time is wasted in trying to conduct a class in each subject for all or most of the grades. It will be found entirely feasible to combine the arithmetic classes of two grades as of the fifth and sixth grades, and of the seventh and eighth grades. Time can also be saved in the rural schools by giving arithmetic on three days in the week for longer recitations and alternating it with some other subject for the remaining two days. The following month arithmetic is given two days and the alternate subject for three days of the week. Time required in calling one recitation per day is thus saved. It is also a good practice to call the advanced classes first. The pupils of these classes may then be working on the board in view of the teacher while the next class is reciting. Use of such schemes will be found very repaying to the teacher. If necessary, explanations can be made to

the parents that no subject is slighted by this process but is really given more attention, because the teacher and the pupils will have more time to devote to actual work.

100. General Suggestions.

1. Teach by precept *and* example. The teacher must be careful to do just what he requires from his pupils.

2. Complete the lesson assignment from day to day.

3. Leave no work unsettled at the end of a recitation unless it be a question which it is desired that the pupils should consider further. In such a case it should be cleared up at the beginning of the next lesson.

4. Require good English but do not stop an interested boy in his explanation of a problem to correct his grammar. Permit him to complete what he has to say first and then when the arithmetic work has been settled direct his attention to the mistake in English.

5. Learn to teach without the text as far as possible. This can be done with most of the oral work. The ability to dictate examples to a class working at the boards without reference to the text, gives the teacher confidence in himself and the pupils confidence in the teacher. It also gives the teacher opportunity to see the work of every pupil and gives him general strength and resourcefulness.

6. Do not be an extremist or faddist. Let "Evolution and not revolution" be the slogan.

7. Be continually on the lookout to improve by adopting and adapting any new practical ideas.

8. Always look for the **big idea**. Be careful never to confuse the non-essential with the essential.

VIII. EXERCISES

1. Explain and illustrate the difference between the language and the mode of thought in arithmetic.

2. Explain the necessity of not confusing the language and mode of thought.

3. Give two illustrations showing how the language may be taken for the mode of thought.

4. Suggest two examples of incorrect use of symbols.

5. What degree of freedom should pupils be given in the use of symbols? Illustrate.

6. Discuss the need of system and orderliness in arithmetical work. Illustrate.

7. What is the teacher's first requisite for teaching system and neatness successfully? Illustrate.

8. What is meant by the unification of arithmetic? Illustrate.

9. Explain how unification will simplify the work and save time. Illustrate.

10. Give three illustrations of inverse processes.

11. Discuss and illustrate each of the suggestions for attaining the unification of the subject.

12. Explain how short-cuts are a form of unification. Illustrate.

13. Make a very brief lesson plan which can be used in teaching.

14. When and how should the next day's lesson be assigned? Illustrate.

15. Discuss the various forms of wasting time mentioned under 98.

16. Suggest and discuss other forms of wasting time than those mentioned in 98.

17. What are the advantages of grouping the work as suggested for rural schools?

18. Suggest a workable plan in which the work of the grades is combined for rural schools as mentioned under 99.

19. Discuss the other suggestions for saving time in a rural school.

20. Discuss each of the general suggestions on methods under 100.

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CHAPTER IX

PRIMARY ARITHMETIC

101. Basis of Early Number Work. Normal children, when free, delight in realizing their ideals through almost any convenient medium of expression, as with paper, sand, water, wood, ready-made dolls, or the written or printed word. The pleasure is in the spontaneous action in anticipating or in realizing a desired result. The pleasure and the educative value cease when the cutting and the painting are imposed as *tasks,* which must satisfy the requirements of a teacher seeking her own *definite, preconceived results.*

Work suggested is not valuable unless it stands the following tests:

1. *That the children feel in it a purpose or satisfaction which arouses their best efforts.*

2. *That it be so adapted to their mental and physical powers that there can be no overstrain connected with it.*

3. *That it unconsciously tends to cultivate good taste in the children.*

4. *That it is work that the children themselves can criticize as they discover its failure to fulfill the purpose for which they designed it.*

All the work in the grade might have been included in an outline based upon the creative and constructive activities of the children. For instance, in the nature-study work, the children are brought in contact daily with real things on the school grounds; in the garden; in their trips

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to typical landscapes, as the lake shore, sand-dunes, swamps, and so on; in their visit to the farm and to the various industrial plants of the city.

The experiments performed are those which help them to answer their own questions, and the effort is to give them continual opportunity for clear, definite imaging and the genuine unlimited expression which image-growth demands.

In literature the stories are selected with care and many of them are dramatized by the children.

The chart of section 102 presents only the things which can actually be made each month for the home, the school community, or the immediate use of the pupils making them.

The exercises are not, however, classified under these heads. It will be noticed that in many cases the power and skill gained in making one article is directly applied in the making of another. This, however, is never a guide to *what* is selected for making, though it may many times determine *how* the needed article is made.

In making the articles each child first plans the work independently, after which the best, or the most convenient, plan is often selected for actual making by the class.

In this work the children are continually planning, making, comparing, judging, estimating, and verifying, and thus the number element comes of real necessity into daily use. The pupils are continually meeting the quantitative questions of "How many?" and of "How much?"

102. Early Construction Work. The chart here shown suggests various forms of early construction work for the different months of the year. In the early fall and late spring much of this work will naturally be replaced by outdoor excursions and activities.

103. Forms of Early Number Work. In the primary grades children need to measure constantly in constructing such articles as are suggested in the chart and in making

simple apparatus in cooking, gardening, and other school activities. They need to count and keep scores of games, to keep account of the books and materials they buy, to learn to tell time, and they soon recognize that numbers and calculations are essential in their work.

In order that they may do this mathematical work intelligently, they must early become acquainted with the standard units of measure in lines, areas, volume, time, weight, and value. For instance, that there are 12 inches in a foot should not be committed to memory as an isolated fact, but the child should have a knowledge of the foot as a factor in his judging power and as a necessary tool for measurement.

104. A Minimum Requirement in Arithmetic. There should be a minimum requirement in arithmetic for each grade, which should be determined by the needs and experience of the children, as decided by the teacher of each grade in coöperation with the teachers of the next higher grades of the school. Very little should be required the first year, but certain facts and processes should be thoroughly functioned in each grade, so that the children may be relied upon to have a growing fund of arithmetical knowledge ready for use.

105. Social Projects. If the children work constantly upon small, concrete social *projects,* there will be demands for measurements and for increasing knowledge of arithmetical operations. The teacher will be responsible that the facts and processes involved in these problems do not present obstacles too great, but that they are sufficiently difficult to require the children to struggle to the full limit of their ability to overcome them.

106. Significance of the Minimum Requirement. Problems arising in the work may take the children beyond the minimum planned for the grade, and many individuals in

the class may voluntarily go far beyond the limit, but it is the teacher's duty to see that all the children have a mastery of the tools and the right habit of solving problems within the scope of the minimum requirement. This does not mean that the children must solve a mass of isolated problems or have unrelated drill, for, in a school where the curriculum is based upon activities, there will be more demands for the solving of problems than there is time for, and drill will come as a welcome aid in overcoming obstacles between the child and the desired goal.

IX. EXERCISES

1. Which of the following activities satisfy the test in 101: making valentines; white-washing a fence (a) alone and (b) with companions; digging dandelions; playing number games; making May baskets?

2. Suggest three child activities which will stand the tests in 101.

3. Explain how the child enters into number activity in planning and carrying out construction work. Illustrate.

4. Consider question (3) for at least four suggestions in the chart under 102.

5. What should be the predominating motive in early number work? Illustrate. How can it be accomplished?

6. What should be the motive and process of learning the units of measure? Illustrate.

7. State the minimum number knowledge a child should have at the end of each of the first three years.

CHAPTER X

PRIMARY ARITHMETIC (*Continued*)

107. Illustrative Lessons. In order to aid the teacher the following three type lessons on early number work have been prepared.

Concrete typical problems might be chosen from any one of the school activities mentioned in Chapter IX, but since linear measurements are so important in the primary grades, three illustrative lessons based upon these measurements have been selected which may be given in any primary classroom, whatever the size of the class and with little or no equipment. The first two lessons would be among the very earliest number (measuring) lessons of the first grade; the third might come at the last of the second or beginning of the third grade.

The first lessons do not represent the highest type of teaching, since they give opportunity for too little initiative on the part of the children. The teacher dominates in the directions and questions, but the children are given full responsibility for the results of their activities. If a real, instead of a hypothetical, lesson were being described, its success might be gauged by the kind and the number of questions the children were stimulated to ask. But even without this value, it seems a worthy type of lesson for a beginning class in mathematics, since the children will apply the knowledge and judgment they gain immediately and independently in many situations arising naturally in their work and play.

Throughout such a lesson the word "foot" would be associated with the idea (length) it represents. The children would write the word under each line a foot long which they drew. But the word-association process would not be allowed to distract the children's attention from the central point of the lesson.

In each illustrative lesson the quotation marks indicate the teacher's question or direction.

108. Lesson 1.

Teacher's purpose:

(a) To teach the foot as a linear unit of measure.
(b) To help each child to find an approximate measure (his own height) which he can use in judging distances and the height of objects.

Child's purpose.

An interest in activity which at this age is an *end* in itself—merely a game interest.

Lesson.

Time, 25-minute period. Number in class, 20 as a maximum.

Exposition of lesson.
Teacher asks:

"Who is the tallest child in the class?"
(One child is requested to pick out the tallest child with the result that *John* is chosen.)
"How many agree with the choice made?"
"How can you prove that John is the tallest?"
"How tall is he?"
"Who knows how to find out?"

109. Descriptive Detail. As the children have had kindergarten experience, the foot ruler is suggested, and the children are each given foot rulers to examine.

They are then asked to select red, blue, or yellow strips of paper, *exactly a foot long,* from those hung among longer and shorter ones on a wall of the room. After choosing they measure and verify their judgments.

One child is asked to draw upon the blackboard a perpendicular line a foot long—(like the teacher's); to draw a horizontal line a foot long (like the teacher's). After these lines are erased, the children are all asked to go to the board and draw, without rulers, a perpendicular line a foot long and a horizontal line a foot long under it.

Children measure and correct lines, and note how much more accurately they have drawn the horizontal lines than they have the perpendicular lines. Then all the children in the class are asked to hold their hands exactly a foot apart, one hand opposite the other; then one hand above the other a foot apart, while the teacher quickly measures the distance between the hands with a ruler.

The number of devices used will depend upon the teacher's judgment of the need of the particular class, and what he considers the best use of its time and energy. With a very slow class, a lesson might end at this point.

110. Possible Further Development with a Strong Class. With a more mature group of children the teacher would return to the problem presented in the first part of the lesson and perhaps next ask the children to hold up the strips of colored paper that they had found and measure which were just a foot long.

"How many of these strips a foot long would it take to make a strip as long as John is tall?"

In spite of the day's experience, the teacher will possibly have answers indicating that John is anywhere between two and ten feet tall.

"How may we measure more easily than with the strips to find out John's height?"

At this point it is expected that some child will discover a planed plank marked off into foot spaces, placed within reach by the teacher who has foreseen this need. John is asked to stand with his back and heels firmly against this board. It is found that he is 4 feet tall, and *more.*

The lesson ends by giving the children, for seat work, the following directions:

"Make four strips, each one foot long, and as wide as your ruler, and bring them to class tomorrow. Also bring your ruler."

111. Lesson 2.

Teacher's purpose:

(a) To review and impress the foot unit.
(b) To teach the new fact that there are 12 inches in a foot.
(c) To continue the problem of finding the height of the children.

Children's purpose:

To continue the pleasurable activity—(during this lesson a definite problem will take form in their minds).

Lesson:

Time: 25-minute period. Same number of children in class (20).

Exposition of lesson:

Children help the teacher examine strips of paper made at home, and quickly select accurate ones. Each child holds up his correct strip. Great accuracy is not to be expected from all of the class after one trial. Effort is approved. Careless work is disproved and discarded.

Children then choose, from colored strips hung on the wall, until each child has four strips, each 1 foot long.

Teacher directs:

"Hold up the strips that are right which you yourselves made."

"Tell how many strips you had to choose in order to have four correct ones."

"Who would like to have a strip just as long as he is tall?"

"If each one made another strip in June we could find out just how much each one has grown this year."

"How can you find out just how tall you are now?"

"Yesterday we found that John was 4 feet and *more*."

"How much more?"

"How can we find out?"

"Please take your rulers and examine them carefully."

"Tell what these small divisions are." (Teacher points out inch spaces.)

The teacher then proceeds to teach the inch-unit idea in much the same way that the foot-unit was taught before.

"How many inches are there in a foot?"

The class counts as one child points.

Class claps as one girl points to the inches on the ruler.

With various devices the class repeats the counting exercise until all of the children have gained the knowledge that there are 12 inches in a foot. The lesson ends with the following directions for seat work:

"We shall begin today to make our long strips to see how tall we are now."

"You may each tell a way in which it could be done."

If no child suggests a better way, the teacher asks the children to follow her way, as follows:

"Divide each of your four correct strips into one-inch spaces."

"Paste the strips together, allowing 1 inch on the end of each strip to overlap."

"When you have pasted the four strips together, measure your long strip with a ruler."

"How many inches short of 4 feet is it?"

"Any one who can, may make a short strip just long enough to make the long strip just 4 feet in length."

"All bring their long strips to class tomorrow."

"Those who make the short strip bring it to class also."

112. Lesson 3. The following number lesson was given to a group of fifteen third grade children. The problem arose when the children were getting ready to plant their gardens, and were experimenting to find in which soil (sand, loam, or clay) the seeds would grow best. One child suggested that light had much to do with the growth of seeds; another, that the amount of water which the soil holds makes a great deal of difference. This led to the desire to find out which soil held the most water, and which held water the longest. As each child wanted to perform his own experiment, it was suggested by the teacher that they would need to use the same amount of soil for the tests in order to make their comparisons of any value. Among the suggestions for measuring the soil was that of one child, who said, "We could each make the same sized box at home and bring it to school tomorrow." Although this was a round-about way of getting a measure, there was no objection made by the class, and the teacher, seeing opportunities in the suggestion for teaching certain things of value to them, accepted it, and asked the children to make boxes at home from wrapping paper or cardboard, of any shape which seemed convenient—the only requirement being that each box should hold exactly 4 cubic inches. The next morning, before the science lesson, the teacher gave the number lesson which follows:

Points in the lesson which seemed to the teacher important:

(a) To encourage initiative and independent judgment.

(b) To make children responsible for the preparation of their own materials.

(c) To teach economy in the use of materials.

(d) To make clear the fact that a tool of measuring *must* be *exact* and should be convenient.

(e) To review work in lines, areas, and cubical contents, by pre-

senting a *new* problem similar to others the children have previously solved.

Possible reasons why the lesson seemed important to the children:

(a) Because it was part of an interesting experiment answering a question they themselves had asked, and by a method of measuring they had suggested.

(b) The activity itself involved in such an experiment finds a sure response in the interest of children.

113. Subsequent Lesson:

Time: 25-minute period. Number of children in class, 15.

Exposition—Number Lesson.

The first 5 minutes were spent in examining the boxes made at home.

Three shapes were brought by the children.

The teacher then presented her set of four boxes, each of which held 4 cu. in.

The class selected two of the best which they had made and added two made by the teacher of different shape.

They found

Box No. 1, $4 \times 4 \times 1$, with open area at top of 4×1.
Box No. 2, $2 \times 2 \times 1$, with open area at top of 2×2.
Box No. 3, $2 \times 1 \times 2$, with open area at top of 2×1.
Box No. 4, $1 \times 1 \times 4$, with open area at top of 1×1.

The children were asked to pick out the box of the shape most convenient for use in measuring the soil. They unanimously chose No. 2 for this purpose. They were then asked to select the box which could be made from the smallest piece of paper and hold 4 cubic inches. Each child passed to the blackboard and recorded his judgment under his own name, after looking carefully at the boxes. Three children

thought box No. 1 took the least paper, nine thought box No. 2 took the least paper, two thought box No. 3 took the least paper, and one child box No. 4.

The boxes were then unfolded and their areas compared, as follows:

Box No. 1, area $3 \times 6 = 18$ sq. in.
Box No. 2, area $4 \times 4 = 16$ sq. in.
Box No. 3, area $5 \times 6 = 30$ sq. in.
Box No. 4, area $9 \times 9 = 81$ sq. in.

The children were much impressed when they discovered that four boxes like No. 2 could be made out of a piece of paper large enough to make box No. 4, and still enough paper be left to make box No. 4 with the exception of strips to bind it together.

Box No. 2 was later used by the class in the science experiment.

Copper boxes, each holding 16 cu. in., of various shapes and open areas, can be made in a tin shop, and are exceedingly useful in experiments in evaporation and condensation. Children place the boxes, filled with water, under varying conditions of heat, and note the different rates of evaporation, and compare the open surfaces in their attempt to solve the problem. This work demands both accurate observation and much arithmetical computation.

X. EXERCISES

1. Why is the work on linear measurement so important? Illustrate.
2. How does the work on linear measurement connect with that on areas and volumes? Illustrate.
3. What are the essentials of a good early number lesson? Illustrate.
4. Discuss and criticize lesson 1. Write out the description of a similar lesson of some other phase of early number work.
5. Consider lesson 2 as in question 4.
6. Consider lesson 3 as in question 4.
7. Outline a lesson in construction embodying the idea of economy as well as measurement in the making of any of the following articles: drinking cup, picture frame, wall pocket, envelope, May basket, or colonial cap.

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CHAPTER XI

ADDITION AND SUBTRACTION

114. History. The difficulties encountered by the ancients in performing the simplest computations in addition and subtraction can best be appreciated by a consideration of the two following operations:

(1) MCCLIX + CCLXIV + CCCLXVII.

(2) MMCCLXXVII — MDCCXXXIV.

The Hindu number-system and abical reckoning* completely revolutionized certain arithmetical processes. Under this influence addition, subtraction, and multiplication became practically what these operations are today. A vital difference was that the operations were carried on from left to right, a method which continued until about 1600 A. D. This method of course seems very awkward to us who are acquainted with the simpler scheme of beginning at the right. The following is an illustration of the old process in addition:

423
165
274
752
862

Thus, the sum of the first column to the left is first written down, $4 + 1 + 2 = 7$; then the middle column, $2 + 6 + 7 = 15$, from which the "1" is carried over to the "7" making "8" which is written below the "7"; then $3 + 5 + 4 = 12$, from which the "1" goes over to the "5," etc., thus leaving 6 to be written below the 5, and giving as a final result 862. In actual practice the 7 and 5 were never retained but erased, which was a simple matter by

* Reckoning with the aid of the abacus.

the use of the abacus, or the sand numerals. Results were also always written above the given numbers instead of below as we do. The first two processes of subtraction outlined in **123** were used during these early years.

Before the introduction of the idea of place value, numbers were indicated by the additive value of the symbols written in order as in the Roman notation of today. Hence, numbers to be taken additatively were written one after the other without any connecting signs. Subtraction was indicated by placing a dot over the subtrahend. The words *plus* and *minus* arose sometime during the 13th or 14th century; the first record of the symbols + and — occurs in an arithmetic published by JOHN WIDMANN in 1489.

115. The Four Fundamental Operations. The main task of the teacher during the children's first four years of number work is to give them a facility in the use of the four fundamental operations with integers. The poor results accomplished during this time are most discouraging to teachers, and not the least so to teachers of the higher grades. Intelligent teachers who really desire to give the children under their care ability to perform readily the fundamental operations with integers, will not find this a burdensome matter providing they study the problem seriously and set about achieving their object in a systematic manner with a definite purpose always before them.

In the first place, the main business of the child now is to learn the *process* of the operation and not the *reasons* underlying it. This must not be learned blindly, however, as will be seen when subtraction and division are considered; but the **big thing** is teaching the child the *art* of computation, not its *science*. Let the teacher's slogan at this time be an appeal to the habituation, and very little to the rationalization, side of the child's nature. The teacher's questions during the child's early number work

must be almost wholly concerned with the "how" and the "what," and very little with the "why." An analysis of the teacher's task in the early number work shows that it is twofold: *memorization* of the various combinations and their *application* in computation.

116. The Teacher's Duty Here. The work in the textbook is so arranged that the various difficulties of any subject are taken up singly and in such gradation as to produce the least confusion to the pupils. New work is generally based upon what has preceded under that particular section,—as in the case of addition. Most texts of the present day do not take up all of addition before subtraction, multiplication, and the like or all of subtraction before multiplication, and so on. The arrangement of the work in any topic of the subject, such as subtraction, depends upon the related topic, in this case addition, which has preceded it. Therefore, the text is recommended to the teacher as a general plan to follow. As an aid to those teachers required to use certain texts that do not follow this advantageous idea of unification, and in order that all may realize more fully the importance of the order of introducing the various new elements in each subject, and, further, because a particular order of presentation seems more advantageous and logical than others, this matter of arrangement will be given some attention in the following lessons as they deal with the presentation of the four fundamental operations. Space limitations make it impossible here to go into detail, however, concerning the question of arrangement in these lessons and the teacher is accordingly advised to study carefully the text he will teach from this standpoint of advantageous arrangement for every topic as it will come up for consideration.

117. Combinations in Addition. While the nine digits can be arranged one after the other in 362,880 different

ways, only 45 combinations can be made from these same digits by taking two at a time. This means that there are only 45 sums of two numbers as, $7 + 2$, which must be learned; $7 + 2$ and $2 + 7$ are, of course, treated as one combination. The first requisite for accuracy and speed in addition is a thorough acquaintance with these 45 combinations. It is necessary that these combinations be learned so thoroughly that the sum of any two single digit numbers equals a third number as inevitably and naturally as a combination of letters spells a word; $7 + 2$ must be 9 just as spontaneously as c-a-t is cat.

Teachers who have had experience in teaching addition to children realize the absolute necessity of a thorough acquaintance with the 45 combinations. Good teachers throughout the land are daily demonstrating the fact that normal children may acquire this knowledge at a comparatively young age; as compared with the formation of words from letters including the pronunciation of these words, the formation of these 45 sums is a very small undertaking. A definite *goal* and systematic procedure toward that objective on the part of the teacher are the necessary factors.

118. Advantageous Order of Taking Up the 45 Addition Combinations. The order in which these combinations are taken up is an important matter. Not many years ago texts contained the tables of 1's, 2's, 3's, etc., which were memorized by the children in that order. Children who had thus learned the nine tables could quickly repeat any one table mechanically but were usually unable to apply them readily in the addition of columns of figures. When one of these children met with $4 + 7$ in the addition of a column he was likely to say $4 + 1$, $4 + 2$, etc., until he came to $4 + 7$, or, he might, if alert, turn the numbers around and say $7 + 1$, $7 + 2$, $7 + 3$, $7 + 4$. In either case the loss of time is greater than in the still older way of

counting upon the fingers, thus, 8, 9, 10, 11,—a finger being turned down for each unit added. Each combination must be learned as an *isolated fact* in order to be useful in the process of addition. If $4 + 7$ is learned by itself there will be no suggestion to "run down the table" until the needed combination is reached. In going up a column of figures the sums will then be read in a manner similar to that of combining letters into words.

1+1	1+2	1+3	1+4	1+5	1+6	1+7	1+8	1+9
2+1	2+2	2+3	2+4	2+5	2+6	2+7	2+8	2+9
3+1	3+2	3+3	3+4	3+5	3+6	3+7	3+8	3+9
4+1	4+2	4+3	4+4	4+5	4+6	4+7	4+8	4+9
5+1	5+2	5+3	5+4	5+5	5+6	5+7	5+8	5+9
6+1	6+2	6+3	6+4	6+5	6+6	6+7	6+8	6+9
7+1	7+2	7+3	7+4	7+5	7+6	7+7	7+8	7+9
8+1	8+2	8+3	8+4	8+5	8+6	8+7	8+8	8+9
9+1	9+2	9+3	9+4	9+5	9+6	9+7	9+8	9+9

Although the combinations should be learned as separate facts it is well for the teacher to have some definite order of selecting them. Otherwise difficulties will arise, as in all cases where there is a lack of order. A reference to the above table will show that the sums contained between sets of diagonals as have been drawn, are all the same. One scheme for selecting the combinations to be learned is as follows: Begin at the upper left-hand corner and learn the combination above and to the left of the first diagonal, namely, $1 + 1$. Draw the next diagonal and learn the combinations contained within the two lines, and so on through the list. In this manner, small numbers and small sums

are used at first and both numbers and sums increase simultaneously as the child grows more practiced. It should also be noted that all such dual combinations as $4 + 5$, and $5 + 4$, are included between any one pair of diagonals, and hence would be taken up simultaneously by such a scheme, as has been previously recommended. Such plans depend upon the ingenuity of the teacher.

There are many plans of selecting the combinations which will likely work as well providing always that each combination be learned as an *isolated fact* and that some system be used in moving from the simpler to the more difficult. Various devices for eliminating some of the drudgery of the drill work in fixing these combinations in the mind of the child have already been considered under Chapter VII.

119. Order of Work in Addition. Begin the work in addition with a few simple combinations like $1 + 2$, $2 + 1$, $1 + 3$, $3 + 1$, $2 + 2$, etc., as previously suggested. Make this work oral and introduce no new combinations until those already under consideration are thoroughly mastered. Use a large number of simple concrete problems adapted from the experiences of the pupils, as in the following:

"If John has 3 cents and earns 2 cents more, how many cents will he then have?"

The value of this work will be directly proportional to the vim and alertness of the teacher in giving it. Let it be borne in mind, that this oral work should not cease when written work is taken up, but the two, oral and written, should progress hand in hand. The teacher should continue to use all of the combinations which have been previously learned, after new combinations are taken up. Let him pay especial attention to those combinations giving the pupils the most trouble, without neglecting any to such an extent that they later become troublesome.

The following order is recommended for written work:

i. Two or three numbers whose sum is less than 10:

	3
3	4
4	1

ii. Short columns of numbers with one and two digits, requiring no carrying:

23		4
2	34	32
4	21	13

iii. Numbers containing zeros:

30	6
14	20
21	12

iv. Numbers of two digits where carrying is required:

	42
56	28
35	15

v. Longer columns with numbers of two and short columns with numbers of three digits.

vi. Finally columns of any length and numbers of any number of digits.

120. Difficulty of "Carrying" Process. The chief difficulty that arises in this early work is that of carrying,—a process which it is absolutely essential that the children be

able to use accurately and quickly. The idea of 10 ones in each ten and of 10 tens in each hundred must be reviewed thoroughly in connection with the introduction of the new idea of carrying.

121. No Grouping by Tens at First. Let the teacher bear in mind that during the first four or five years the process should invariably be that of adding in the exact order of the numbers in the column and *never* by the selection of groups of numbers making sums of 10 or by some other similar schemes. This method of addition has its proper place, but not in the first years, when the child must learn this process of *reading sums of numbers.*

122. Short Cuts. When the children have learned thoroughly to add by combining each digit number with the sum already found, they can be introduced to a few of the short cuts used by those who are especially quick and accurate. One of the schemes most used is that of picking out groups of numbers which add up to 10 in each column. For instance, in the accompanying example in glancing up from the bottom the following groups whose sum is 10 are found in the right hand column: $5 + 5$, $4 + 6$ (8 passed over), $5 + 5$, $3 + 7$; the 8 with the remaining 4 gives 12; this and the 4 additional 10's make 52 altogether.

264
357
383
945
465
168
786
664
245
325

If a figure repeats itself several times in a column a good scheme is to count up the number of times it occurs and then multiply the number of times it occurs by its value; for instance, if, in the second column, the $2 + 4$ is considered as a 6 and the $4 + 8$ farther up is considered as two 6's, then there are seven 6's in that one column besides an 8 and a 5. There are, however, some experts on addition

who can do more accurate and faster work by carrying the sums right along rather than by making use of this grouping of numbers. Whichever method fits the individual best is, of course, the one to follow.

123. Methods of Subtraction. Analogous to the case of "carrying" in addition, is that case in subtraction in which a digit number in the subtrahend is larger than the corresponding one in the minuend, as,

$$\begin{array}{r} 53 \\ 28 \\ \hline \end{array}$$

The following three methods are the ones most usually employed in performing this subtraction:

i. 10 is added to both minuend and subtrahend, thus making the subtraction 8 from 13 and 30 from 50.

ii. 10 is taken from the 50 and added to the 3 making 13 when there is 8 from 13 and 20 from 40.

iii. Subtraction is treated as the reverse of addition and the question becomes, "How much must be added to 28 to give 53?"

124. Example. The Austrian Method. In the case of the accompanying example the result is so simple that an explanation is superfluous, but the one above should be considered quite carefully. Since 8 can be added to nothing to make 3, the question is, "What added to 8 gives 13?" to which the answer is 5,—the first digit to the right in the remainder. In the addition of the 5 and the 8 in the units column only the 3 was written down above and the 10 carried over to be added to the 20 and the tens in the remainder. Hence this carried 10 and the 20 make 30 to which the tens below the line are to be

$$\begin{array}{r} 47 \\ 23 \\ \hline \end{array}$$

added to make 50, and the question becomes, "How many added to 30 make 50?" or more simply, "How many added to 3 make 5?" This method is referred to as the "Austrian," "continental," "adding to," or "making change," method. The simplicity and naturalness of the method is shown by the fact that it is universally used in making change. It may also be said to be the natural method since it is adopted by almost all self-taught arithmeticians. It is the simplest method for the children who have learned no other, and those pupils who use it have been found through tests to make better progress than children who use one of the other methods. Subtraction is the reverse of addition and should be so taught as to give the children this number relation. While the Austrian method has many advantages over the other methods the **big thing** is that it connects subtraction with addition thus giving the children an idea of number relations by unifying the various processes of arithmetic. There are no valid objections to the Austrian method, and it should by all means be given to children who have learned no other process. The advisability of requiring all the children of a school system who are already familiar with one of the other methods to change is questionable. A gradual change, however, as here suggested is not only possible, but highly practicable.

125. Order of Subtraction. Introduce each of the following types, or examples of subtraction as soon as the pupils are thoroughly acquainted with the corresponding types of addition previously outlined:

i. A single digit number taken from another single digit number:

6	8	9
4	3	2
—	—	—

ii. A single digit number taken from a two digit number giving a remainder less than 10. A two digit number taken from another two digit number, each digit of the subtrahend being less than the corresponding one in the minuend:

15	13	29	43	63
6	5	9	21	42

iii. Same as (ii) with zeros in subtrahend:

46	31
20	20

iv. Some digits in minuend less than corresponding ones in subtrahend:

54	48	63
36	29	27

v. Same as (iv) with three digits in numbers.

vi. Several digits in each number.

By thus coördinating subtraction with addition the children will learn that 10 is 6 more than 4 and that 10 is 4 more than 6 at the same time that they learn that $4 + 6 = 10$. Simple problems such as the following, of which many should be given daily, will be found a great aid at this time:

"John has 3 cents now; how many must he earn to have 7 cents?"

"How many cents must John earn to have 8 cents, if he has 5 cents now?"

"Mary is 7 years old. In how many years will she be 11 years old?"

"John has 5 marbles and James 9 marbles. How many more marbles has James than John?"

With a little practice the alert teacher will find that any number of these simple problems can easily be made up and that, in time, examples to suit the occasion will suggest themselves. These problems will gradually increase in difficulty, since the numbers used are, of course, to be those which have been and are being learned. Each type of addition example, together with its inverse in subtraction, must be thoroughly understood before the next is taken up. And thoroughness can only be accomplished by constant and untiring drill day after day. It is also well to include the addition of two numbers occasionally to disabuse the pupils of the idea that only two numbers always mean subtraction.

126. Checks. Unfortunately no simple checks in addition are available. The one most commonly used, that of adding both up and down the columns, is as useful as any. If these two additions vary, a third addition at a later time will generally confirm one of the first two.

127. "Casting the Nines." The "casting out of the nines" is a somewhat shorter test but there are certain cases of errors which it does not disclose; hence it is not infallible. This test is based upon the fact that *the remainder obtained by dividing a number by nine is the same as that found from dividing the sum of its digits by nine.* Any number of three digits, which is sufficient for explanation, can be represented by

$$100a + 10b + c. \quad (1)$$

These can be rearranged into the form

$$[99a + 9b] + [a + b + c,] \quad (2)$$

of which the number in the first bracket is divisible by 9 and hence the number (1) divided by 9 will have the same remainder as $a + b + c$ divided by 9. The remainder found from dividing $a + b + c$ by nine is technically called the

"excess." If the number in (1) above is added to another as

$$100m + 10n + k, \quad (3)$$

the sum will be

$$100(a + m) + 10(b + n) + (c + k), \quad (4)$$

or written in the form (2)

$$[99(a + m) + 9(b + n)] + [a + b + c + m + n + k], \quad (5)$$

of which, again, the part within the first bracket is divisible by nine. Suppose the excess of $a + b + c$ is "E" and of $m + n + k$ is "e," then the excess of $[a + b + c + m + n + k] \div 9$ must be the same as of $[E + e] \div 9$. A short example must suffice. The excess of the first is 5 as the sum

8456	—5	22
1873	—1	19
4509	—0	23
3674	—2	16
18512	—8	18512

of the digits is 23, giving a remainder of 5 when divided by 9. The excesses of the other numbers are also placed at the right. In place of adding the digits of each number we note the presence of 9's as $4 + 5$ in the first, $1 + 8$ in the second, $4 + 5$ and 9 in the third, etc. The excess of the sum is 8, the same as the sum of the excesses added; hence the addition is likely to be correct. Had the sum of the excesses of the numbers been over 9, as 14, its excess would then have been the one to have been compared with the excess of the sum.

128. Other Checks. It may be suggested that a form of addition much used by those liable to be interrupted while adding, is to write down each sum to the right, as was done above, and then to add these partial sums.

Subtraction can be verified by either casting out the nines, subtracting the remainder from the minuend, or by adding the remainder to the subtrahend.

129. Number Relations. It is often possible to recast the given numbers so that their sum or difference can easily be found without the use of pencil and paper. For instance,

$$49 + 37 = 50 + 36 = 86;$$
$$34 + 25 = 35 + 25 - 1 = 59;$$
$$28 + 13 + 46 = 30 + 17 + 40 = 70 + 17 = 87;$$
$$54 - 26 = 56 - 26 - 2 = 30 - 2 = 28;$$

etc. The use of these and similar schemes for bringing number work within the range of mental processes gives excellent practice in seeing number relations, an ability which constitutes real arithmetical growth, as has previously been pointed out.

XI. EXERCISES

1. Give a historical résumé of the development of addition and subtraction.

2. What should be the object of the first four years in number work? Is the science or the art of arithmetic to predominate at this time? Illustrate.

3. Discuss fully the method of teaching the addition combinations. What is the big essential? Illustrate.

4. Should addition be "completed" before taking up subtraction? Why?

5. Explain how to teach the carrying process to a class.

6. Discuss the nature of the work in addition during the early years.

7. Discuss the nature of the work in addition during the last two or three years.

8. Explain the principle of grouping the digits for rapid addition.

9. Apply the principle of grouping to the following:

3681	2893	4571	6582	7461	5590
5283	1074	2906	4517	6431	7319
7326	6105	1674	5710	5231	5163
6412	5261	3467	3531	3657	4681
5372	6301	3584	5216	7351	9216
3216	4163	5418	4136	1485	8215
4213	7183	5173	4257	2163	4182
3415	9316	1346	2516	6132	1547

10. Suggest at least three examples suitable for class-drill in grouping.

11. Recast the following into examples sufficiently simple for mental work:

$28 + 72$	$57 + 29$	$99 + 23$	$64 + 27$
$25 + 57$	$34 - 27$	$65 - 28$	$29 + 43$
$27 + 42 + 13$	$54 + 32 + 21$	$63 + 18 + 42$	$48 + 32 + 17$
$698 + 27$	$253 + 184$	$697 - 27$	$125 - 48$

12. Suggest at least six examples similar to the above for use in a grade class.

13. How can work in recasting examples, as suggested for 11 above, be made effective in grade classes? Illustrate.

14. Explain by examples each of the three principal methods of subtraction.

15. State the advantages of the Austrian method. How does a clerk make change?

16. Perform the following subtractions by the Austrian method:

4536	54712	84613	63125	3628
2734	36550	65435	24547	1736

17. Suggest the order for taking up subtraction.

18. Suggest a series of four examples for subtraction suitable for a grade class.

19. Suggest at least three problems each, for beginning addition and subtraction.

20. Explain the idea of casting out the nines.

21. What are the usual checks for addition? Apply each to three examples in 9.

22. What are the usual checks for subtraction? Apply each to three examples in 16.

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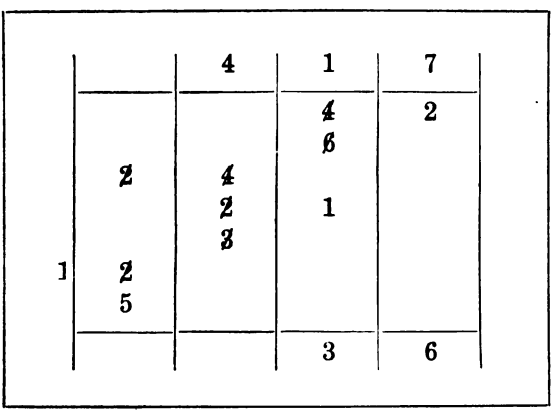
CHAPTER XII

MULTIPLICATION AND DIVISION

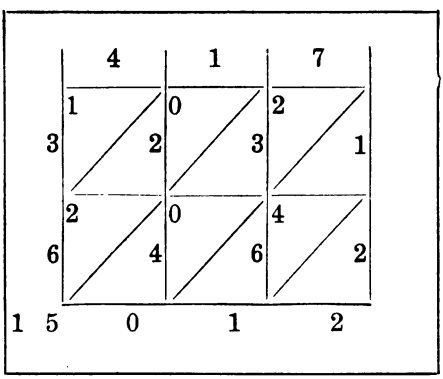
130. History. To try to multiply or divide MMDCXV by CLXXIV would, today, be given up as impossible. Yet until the invention of the number-system embodying the idea of place value, all operations had to be carried out with such symbols. Multiplication was a feat for the learned while division was almost an impossibility. With the ad-



vent of the abacus, multiplication became the simple process here illustrated of finding $417 \times 36 = 15012$.

Products of each pair of digits are first set down and later combined and replaced by a single counter in the appropriate column. Often, a digit goes over to the next column as in the second column of the illustration. These

combinations take care of themselves in computations with an abacus containing 9 equal-valued counters for each column. The gubar numerals, also, instituted quite a change in the process of multiplication. As the sand tablets upon which the work was performed could hold only few of the necessarily large figures the partial products were not retained for later additions as they are today, but were combined as the work progressed so that only the original numbers and the final result remained at the end. These



processes are usually referred to as the "scratch" or "galley" method.

Another early form was the so-called "latticed multiplication," as shown in the accompanying illustration of the multiplication of 417 by 36. The Egyptians multiplied from right to left; the Greeks and Hindus from left to right.

Division, prior to the coming of the complete Hindu number-system, was a sort of repeated subtraction which excluded our modern idea of quotient. As an explanation of this method of division is too complicated to be discussed

in these pages, those further interested in its study are referred to the arithmetical histories in the list on p. 17.

The use of pencils and paper in computations completely revolutionized the processes of multiplication and division. The smallness with which figures could be made with a pencil upon paper made it possible to retain all of the work, while erasures were correspondingly difficult and consumed time. Retention of all the figures is also a great aid in correcting any errors which may have crept in. Thus mathematical computations could be made more accurate by the use of pencils.

131. Arithmetical Symbolism. The Hindus indicated multiplication by writing the abbreviation for product after the last factor, and they indicated division by placing the divisor below the dividend. Our present day signs for multiplication and division originated during the 17th century; $\times$ by OUGHTRED in 1631 and $\div$ by RAHN in 1659. HARRIOT also used $\cdot$ as the sign of multiplication and LEIBNITZ: for division at this time. It is very unfortunate that the signs $\frown$ and $\smile$ suggested by Leibnitz for multiplication and division were not accepted as they show so clearly the relation of the two processes; namely, that each is the reverse of the other.

132. Multiplication. What was said in the last lesson regarding the four fundamental operations applies to multiplication and division as well as to addition and subtraction.

There are really only 36 combinations to be learned in multiplication, since no time need be taken up with those containing a factor 1, as 1×2, 1×3, etc. The pupils must be so grounded in these 36 combinations, however, that work with them becomes a sort of reflex action in much the same manner as reading, so that 3×7 is 21 as spontaneously as d-o-g is dog.

In order to achieve the desired degree of skill, each combination must be learned as a single isolated fact apart from all the other combinations so that there will be no temptation to "run down the table" until the needed combination is reached. One aid to securing such a command of the combinations by the children is by the inversion of the same numbers, as for example, the use of 7×3 as often as 3×7. For suggestions on teaching such combinations, see Chapter VII on drills.

Multiplication should be taught as an abbreviation of addition. The fact that multiplication by a fraction will not be in accord with this statement, as a number can not be added to itself $\frac{1}{3}$ or $\frac{2}{3}$ of a time, in no sense invalidates this common-sense manner of looking at the multiplication of integers.

It is not wise in these early lessons to enter into a philosophical discussion of the abstractness of the multiplier. Indeed, this need never engage the attention of the class. As a matter of fact it is numbers which are being dealt with, and not concrete things.

133. Order of Work in Multiplication. Taught as an abbreviation of addition each new step in multiplication should follow the corresponding step in addition. Most texts on elementary arithmetic combine the work of the four fundamental operations with integers, and can, therefore, be followed without modification by the teacher. Because not all teachers are so fortunate as to be using such texts and because of the vital importance of the matter of order in approaching the work, the following general outline for multiplication is suggested.

134. Introductory Step. Introduce the work by the addition of two 2's, two 3's, three 2's, etc., after which let the teacher explain the shorter process—multiplication—for finding such sums. One or two multiplication combina-

tions can be learned during this first lesson, after which examples in multiplication should be mixed daily with exercises in addition. The daily work must include numerous simple problems, such as the following:

"How many cents will John earn in two days if he earns three cents each day?"

"May put two cents in her savings bank on Monday, Wednesday, and Friday of last week but none on the other days. How many cents did she put in her savings bank last week?"

It might be added, that these little problems should deal as much as possible with experiences in the daily life of the child. By thus humanizing the work, that element of "motivation," the hue and cry of all modern teaching, is introduced, and a real and fresh interest is infused into the daily work.

135. Steps Accompanying Addition. The following order accompanying addition is recommended:

i. Combinations yielding products less than 10, as: 2×2, 2×3, 3×3, 4×2, etc.

ii. Numbers of two digits in the multiplicand and one digit in the multiplier, requiring no carrying:

23	42	Later:	63	91
3	2		3	2

iii. Zero appearing in the multiplicand, requiring no carrying:

40	10	50	40
2	3	3	5

iv. Simple examples involving carrying:

14	26	36
3	3	4

Later examples with zero in products:

24	45
5	4

v. Two digits in multiplier that require no carrying:

34	13
21	32

136. With Two-Digit Numbers. It is necessary for the teacher to explain the meaning of multiplication by numbers of two digits at this time. Multiplication by the *unit's* digit gives the first partial product in *units, tens, hundreds,* and so on. Multiplication by the *ten's* digit gives the second partial product in *tens, hundreds, thousands,* and so on. This paves the way for such multiplications as:

34	32
20	40

Next two digits in multiplier, requiring carrying:

34	64
61	43

It is suggested that the carrying process be taken up slowly so that it may be assimilated thoroughly.

vi. Multiplication of three digit numbers by two digit numbers:

321	524		503	414	302
32	43	Later:	23	20	40

vii. Multiplication with numbers of three digits and more. Variation in the numbers used is henceforth very necessary.

137. Division. Multiplication and division should be taught as two inverse processes; each as the reverse, or the undoing of the other. This idea of reciprocity is most important and ought to be thoroughly understood by the pupils in order to secure the best work later.

Pupils acquainted with the fundamental arithmetical process of reciprocity will not hesitate at such problems as that of finding the number of pencils which can be bought for 12 cents at 2 cents each, if they know how to find the cost of a given number of articles at a known price. As soon as the children can perform the simplest forms of multiplication under (ii) it will be found wise to introduce division by such examples as the following:

"How many 3's are there in 12?"

From this time forth drill on the multiplication combinations should embody such questions as "How many 4's are

there in 12?" as often as "How many are there in three 4's?" The early work in division can be happily introduced by the separation of a number of sticks into groups for which the question would be,

"How many groups of three sticks in each group can be formed from twelve sticks?"

Requesting the pupils to divide 13 sticks into groups of 3 sticks in each group also affords an excellent opportunity for showing the meaning of remainder.

138. Two Forms of Division. Two forms of division are recognized; division by *measurement* and division by *partition*. The former is concerned with such matters as the number of groups of 3 inches each which can be made from 12 inches; the second, with the number of inches there will be in each of the 3 equal lengths formed from 12 inches. It is well for the children to be able to distinguish between these two forms, but the wisdom of requiring them to learn the names of each form is questionable. Many teachers delight in having the children learn that 6 apples can be divided by 2 apples, or that 6 apples can be divided by 2, but that 6 can not be divided by 2 apples. Such philosophical discussions are liable to be as confusing to the children as they are to teachers and would better be omitted.

139. Order of Division. The best results in division will be obtained by following each of the forms in multiplication, as soon as it is fully understood, by the corresponding one in division. When the children have learned certain combinations for multiplication and their inverse for division they should be drilled in both simultaneously thereafter. In no other way will they learn to see so clearly the close relationship between the two; namely, that each process is the undoing of the other.

i. Division of numbers less than 10 giving no remainder:

$6 \div 2$ $8 \div 4$ $9 \div 3$ $4 \div 2$

ii. Dividend of two and divisor of one digit, giving no remainder:

$24 \div 2$ $46 \div 2$ $36 \div 6$ $36 \div 3$

iii. Zero in dividend:

$40 \div 4$ $60 \div 3$

iv. Divisor of one digit requiring carrying, giving no remainder:

$54 \div 2$ $65 \div 5$ $56 \div 4$

Later no carrying but giving a remainder:

$14 \div 5$ $16 \div 5$ $12 \div 7$ $18 \div 8$

v. Carrying and giving remainders:

$742 \div 3$ $567 \div 5$ $542 \div 4$

Short division should be used in all of the above examples and whenever the divisor is a number of one digit. The form $4\overline{)356}$ in which the quotient is placed above the dividend is recommended.

vi. Division by divisors of two digits resulting in no remainders:

$182 \div 13$ $182 \div 14$ $384 \div 12$ $384 \div 32$

vii. Division by divisor of two digits resulting in a remainder:

$256 \div 12$ $427 \div 13$ $345 \div 16$ $576 \div 25$

140. Long Division. As this is the section for introducing long division it is also the time that marks the entrance upon the most serious work in division and that which will require the most careful guidance of the teacher. Even though the pupils may be unable to see the reason involved in the various steps they must be able to carry out the processes successfully in all detail. Selection of the trial quotient will be found by far the most troublesome part. This will require much drill on the part of the pupils and a great deal of patience on the part of the teacher. The pupils should be led to see that the first digit in a

number as 613 will practically decide the digits in the quotient, while the first digit in 487 needs often to be increased by 1.

141. Checks. The process of "casting out the nines" was much used by the Hindus in testing their results because their galley method left nothing at the completion of the computation but the original numbers and the results sought. Even today, when all of the work is preserved for future comparison, multiplication and division can be checked very successfully by this process. For multiplication, *the excess in the product equals the excess of the product of excesses in the multiplier and multiplicand.* In the illustration the excess of 57 is 3 and of 85 is 4. The excess of this product (12) is 3, which is the same as that of 4845, the product of 57 and 85. For division, *the excess of the divisor is multiplied by the excess of the quotient and this product is added to the excess of the remainder.* If the work is correct, the excess of this sum must equal that of the dividend. Thus, in the accompanying illustration, the excess of 35 is 8, of 23 is 5, and their product is 40. To this is added the excess of 19, or 1, making 5. As that is also the excess of 824, the dividend, the division is likely to be correct. The literal equation can be used to good advantage in stating the law of excesses, in which each number is merely replaced by its excess and the operations carried out as indicated for the original numbers. Thus, $P = M \times m$, and $D = Q \times d + R$.

```
 57 —3
 85 —4
---  --
285  12— 3
456
----
4845 — 3
```

```
  35
 ---
824(23
 69
---
134
105
---
 19
```

142. Cautions to Be Observed in Checking. Multiplication can also be checked by reversing the multiplier and multiplicand which, however, becomes awkward in cases such as 3652×64. The teacher must be cautious in applying the check to division by multiplying the quotient by the divisor and later adding the remainder to obtain the dividend. The quotient should never be made the multiplier, since the partial products will then appear in both division and multiplication, and there will be the temptation to copy them from the division in the check, thus missing the error, as in the illustration given.

31	23
824(23	31
79	23
34	79
23	11
11	824

143. Number Relations. The number relations existing in multiplication and division are even more useful in the simplification of computations than those discussed under addition and subtraction. Only a few illustrative cases can here be considered, but the wideawake teacher will constantly be finding others in the regular number work.

In the first place the pupils should be taught that the order of combining three or more numbers by multiplication will not affect the result and that they should select the order giving the simplest multiplication. For instance, the first and third numbers should be multiplied together first in finding the product of

$$20 \times 713 \times 5, \quad \text{or of} \quad 25 \times 315 \times 8.$$

The product of two numbers can often be found much more easily if one is divided and the other multiplied by the same factor. For example, the price of 160 acres of land at \$45 per acre is the same as that of 80 acres at \$90 per acre, or the price of 160 acres at \$22½ per acre is the

same as that of 40 acres at \$90 per acre. In both of these cases the multiplication has thus been reduced to a simple mental operation. Division can also be simplified in the same manner. For example,

$$120 \div 35 = 240 \div 70,$$

which gives a true quotient in decimals, or a true quotient and true remainder, if the latter be divided by 2. Multiplication or division by a number which is an exact divisor of a power of 10 is a very useful number relation. For instance, to multiply or divide by 25, we first annex two zeros or point off two places and divide or multiply by 4, since 25 is $\frac{1}{4}$ of 100. Similar operations will be discussed further in connection with aliquot parts under fractions. To multiply by 279, multiply first by 9 and add to this 30 times itself. Similar multipliers are 328, 546, 287, 654. Use of the above number relations requires a comprehension of the inverse processes together with dexterity in seeing factors, the results of much drill and practice.

144. Some Labor-Saving Devices. The following are a few useful time-saving devices:

1. The product of two numbers, one of which is as much above a third as the second is below it, comes from

$$(a + b)\ (a - b) = a^2 - b^2.$$

$$\text{Thus, } 43 \times 37 = (40 + 3)\ (40 - 3) = 40^2 - 3^2$$
$$= 1600 - 9 = 1591.$$

2. The product of two numbers between 10 and 20 is of the form

$$(10 + a)\ (10 + b) = 100 + ab + 10\ (a + b).$$

$$\text{Thus, } 13 \times 16 = (10 + 3)\ (10 + 6) = 100 + 18 + 90$$
$$13 \times 16 = 118 + 90 = 208.$$

3. The product of two numbers of two digits ending in 5 is

$$(10a + 5)\ (10b + 5) = 100ab + 25 + 50\ (a + b),$$

which for 35×65 becomes

$$(30 + 5)\ (60 + 5) = 1800 + 25 + 50\ (3 + 6)$$
$$= 1825 + 50 \times 9 = 1825 + 450 = 2275.$$

4. Multiplication by 11 is effected by merely writing down the digit to the right then adding it to the one next to the left and so on through with all of the digits. It is the same as adding in

$$\begin{array}{r} 42573 \\ 42573 \\ \hline \end{array}$$

5. Multiplication by a number just under 100 or 1000, as 98, is accomplished by taking 2 times the number from 100 times the number.

6. If the divisor and dividend have a common factor, divide first by this factor. Thus,

$$642 \div 30 = 107 \div 5 = 21, \text{ rem. } 2 \times 6, \text{ or } 12.$$

145. Order of Priority of Arithmetical Signs. This question is taken up more because of the interest connected with it than because of its practical application. The teacher is therefore urged not to give it undue importance. It is also the desire here to call attention to the incorrectness of the statement, "You multiply first."

All operations placed within signs of aggregation, such as parentheses, should be performed first. If the sign of operation is raised to some power, the power-operation takes the next position: $[3 + 5 - 2 + 8 \times 8]^2$ The "of" used to denote multiplication comes after the raising to a

power so that $12 \times \frac{2}{3}$ *of* 6 is $12 \div 4$. The signs $\times$ and $\div$ are of equal importance and the operations are to be performed in the order in which the signs appear. Hence, $12 \div 2 \times 3$ is not $12 \div (2 \times 3)$, nor $12 \div 6$, but is $(12 \div 2) \times 3$, or 18. The signs, $+$ and $-$, are also of equal importance and come last in the scale of order:

$$4+6\div3\times5+8\div2-5=4+[6\div3\times5]+[8\div2]-5$$
$$=4+10+4-5$$
$$=13.$$

XII. EXERCISES

1. Multiply MMDCXV by CLXIV and 2563 by 364 and compare the difficulties in the two cases.

2. Give a short résumé of the early methods of multiplication. Compare with those of today.

3. Apply to multiplication and division the general ideas that the teacher must keep in mind while giving the four fundamental operations.

4. What should be the predominating object in teaching the multiplication combinations? Why? How may this object be attained?

5. Explain what is meant by teaching multiplication as an abbreviated form of addition. What objections are sometimes raised? Elucidate.

6. Discuss fully the last paragraph under 132.

7. When, relative to the other four fundamental operations, should multiplication be taught? Explain fully.

8. Illustrate the introduction of the multiplication process to a class.

9. Consider in detail each set of examples suggested under (i)-(vi) in the order of taking up the work. Suggest at least six examples under each case, which would be good for class board work.

10. Illustrate the various ways in which zeros may enter into the multiplier and multiplicand.

11. Explain fully the meaning of the usual form of multiplication by a number of two digits; of three digits.

12. Explain fully the carrying process. Illustrate.

13. Show that division is the inverse of multiplication. Why should this be taught to the children? Illustrate.

14. Illustrate as to a class how division is to be introduced from this standpoint.

15. How should the division combinations be learned? Illustrate.

16. Consider 7 for division.

17. Illustrate how you would bring the idea of remainder before a class.

18. Explain fully the two forms of division. Illustrate each.

19. What is meant by "divide 12 apples by 6 apples"? "divide 12 apples by 6"? "divide 12 by 6 apples"? Discuss fully to what extent such questions should be considered before a grade class.

20. Consider questions 9 and 10 for division.

21. Explain how long division should be introduced to a class.

22. Explain the checking of multiplication and division by casting out the nines. Illustrate.

23. Apply this test to the following:

$253 \times 43 = 10879$;	$8896 \div 26 = 342$ and 4 rem.;
$217 \times 135 = 29295$;	$2231 \div 29 = 76$ and rem. 27;
$183 \times 62 = 11346$;	$2881 \div 67 = 43$;
$514 \times 137 = 70418$;	$8711 \div 26 = 334$ and rem. 27.

24. What precaution is necessary in permitting the pupils to prove division by adding the remainder to the product of quotient and divisor? Illustrate.

25. Use this check on division examples in Ex. 23. Discuss the result.

26. Perform the following multiplications by the simplest process and explain the principle used in each case:

$25 \times 383 \times 4$	$\$22\frac{1}{2} \times 400$
125×36	316×98
46×12	3562×11
14×16	354×248
75×25	18×22
$12 \times 417 \times 25$	28×4.5
12×24	$125 \times 12 \times 414 \times 2$
124×75	314×22
25×65	416×832
998×68	$3 \times 80 \times 45 \times 2$

27. Suggest at least six similar examples suitable for a grade class. When should this work in simplification be taken up in a grade class and to what extent should it be carried?

28. What is the test for divisibility by 9? by 2? by 3? by 5? by 6? by 15? Illustrate.

29. Prove that a number is divisible by 4 or by 25 if the part made by the last two digits is divisible by 4 or by 25; divisible by 125 or 8 if the part made by the last three digits is divisible by 125 or 8. Illustrate each.

30. If the digits of a number are a, b, c, d, etc., from right to left then the number is divisible by 11 if $a - b + c - d +$ etc., is divisible by 11 or equal to zero. Prove and test.

31. Perform the following divisions by the simplest process and explain the principle involved in each:

i. $27 \div 45$;	v. $4100 \div 25$;
ii. $120 \div 25$;	vi. $1536 \div 27$;
iii. $7000 \div 125$;	vii. $125 \div 75$;
iv. $130 \div 44$;	viii. $648 \div 56$.

32. How can this work in simplification be made effective in grade work? Suggest at least six examples similar to those in question 31 that might be suitable for grade work.

33. Explain the order of priority of arithmetical signs. Apply to the following:

i. $3 \times 4^2 + 7$;
ii. $3 + 16 \div 4 \times 2$;
iii. $4 - 3 \times 6 \div 9$;
iv. $12 \div 4 \times 8$;
v. $[3 + 2]^2 \div 5 - 3$;
vi. $\dfrac{4 \times 5 + 2 \times 4 \div 8}{12 \div 6 \times 5 - 3}$;
vii. $[4 + 3 \times 2] \div 5 + 3^2$.

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CHAPTER XIII

COMMON FRACTIONS

146. History. Fractions proved a stumblingblock to the men of the past ages in developing our counting system, just as they are a stumblingblock to the children in the schools of today. Until recently, computations with fractions were considered a complicated feat and the work was placed late in the texts. Many schemes were devised for overcoming the difficulties, among the earliest of which was that of the Egyptians, as recorded in the AHMES *papyrus*. The fractions in this interesting document have a constant numerator 1 and are hence known as *unit-fractions*. In all computations the fractions had first to be reduced to the sum of unit fractions, as, $\frac{2}{99} = \frac{1}{66} + \frac{1}{198}$. Considerable attention is given in the papyrus to tables of such reductions. For only one fraction, $\frac{2}{3}$, with a numerator other than 1, does Ahmes have a separate symbol.

The fractions of the Babylonians had the constant denominator, 60, in place of a constant numerator and are, hence, known as *sexagesimal* fractions. These sexagesimal fractions were greatly favored by the astronomers of this early period. The Hindus used both the unit and the sexagesimal forms. The Romans used *duodecimal* fractions, that is, those with a denominator of 12. Teachers will do well to consider carefully the scheme adopted by them.

The Roman *as*, a copper coin weighing one pound, was divided into 12 *unciae*.

A separate name was given to each of the 11 subcoins resulting from the division of this unit. These names were

also applied to the 11 forms of fractions which would result through the one denominator, 12. In addition to these *duodecimals,* they also recognized $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{72}$, $\frac{1}{144}$, and $\frac{1}{288}$ by separate names.

There was also great variety in the methods of writing fractions. The Greeks at first wrote the numerator with a single accent, followed by the denominator with a double accent used twice. Thus, according to our present numerals, $\frac{3}{4}$ would have been written 3′ 4″ 4″. The Egyptians wrote only the denominator, placing a dot, or an oval, over it. The Hindus wrote the numerator above the denominator, but without the dividing line, which is a modern notation. Three columns of the Roman abacus were devoted to fractional computations. LEONARDO OF PISA was one of the forerunners in the scientific treatment of fractions. He explained the separation of fractions into unit-fractions and their reduction to common denominators. MICHAEL STIFEL suggested the inversion of the divisor in division. As we come down to more modern times, we find the texts full of complicated rules for the computations with fractions which we are trying hard at the present time to get away from.

147. Informal Introduction. It will be well, in this connection, to re-read what was said in **89** on the elementary work regarding the incidental introduction of fractions. Note that all of this early work is to be entirely objective, and that no formal reference is made to the reduction of fractions to common denominators, etc. The benefit gained later by this incidental introduction is the elimination of the fear and confusion experienced by children in taking up a subject with which they are wholly unfamiliar. The amount that can be done thus incidentally must be left to the judgment of the teacher.

148. Formal Introduction. What was said in Chapter VIII on arithmetical language, applies with especial force

to fractions, as it is in this topic more than in any other, perhaps, that the work degenerates to a mere manipulation of symbols. Special attention must, therefore, be given to the introduction of the subject in order that the pupils may secure a proper grasp of the thought-matter and its symbolism.

In the first place, it must be brought home to the child that, in fractions, he is dealing with isolated things, separate and individual, just as he did in integers; that fractions are separate individuals in their way, only that the unit of measure is smaller than with integers. For instance, a large square can be folded so that when unfolded it exhibits 16 smaller squares, each being $\frac{1}{16}$ of the large square. These 16 smaller squares are each as much an individual square as is the larger one, and the child can count them just as he can count the same number of larger squares. There is nothing new or more difficult in counting by inches than in counting by feet; in counting by quarter inches than in counting by inches.

149. The Introduction to Fractions Objective. The formal, as well as the informal, introduction to fractions should be objective. In fact it is best to include some objective work when introducing each of the four operations with fractions. The paper-folding suggested in most texts is very useful; but to make this really effective, the few illustrations mentioned must be supplemented by a *great number* of similar ones, just as the alert teacher supplements the examples on integers. This work must show clearly such relations as

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}; \qquad \frac{1}{3} = \frac{2}{6} = \frac{3}{9}; \text{ etc.}$$

Linear measure supplies excellent illustrations of fractions as does the paper-folding example. Rulers graduated to fourths, eighths, and sixteenths furnish fine material for

familiarizing the pupils with some of the more common fractions.

150. After the Idea, the Shorthand Way of Writing Fractions. As soon as the idea of fractions has been thoroughly grasped, let the pupils learn the shorthand way of writing them. Give them practice in writing

$$\tfrac{1}{2} = \tfrac{2}{4};\ \tfrac{1}{2} \text{ of } \tfrac{1}{3} = \tfrac{1}{6}, \text{ etc.},$$

as soon as they learn to recognize the truth of these facts. The symbols of each new basic ideal should, however, always follow, and never be presented simultaneously with the ideas, as there is then danger of the latter being covered up by the former.

Reduction of improper fractions to mixed numbers and *vice versa,* may, or may not, precede addition according to the order adopted in the text used. In changing $2\frac{1}{3}$ to an improper fraction, pupils must be led to see that there are six $\frac{1}{3}$'s in 2 and that these are combined with the fraction $\frac{1}{3}$ to make $\frac{7}{3}$. They should not be permitted to multiply 2 by 3 and add the 1 of the numerator, to get the new numerator 7, in a purely mechanical fashion without having grasped the idea back of the process. These ideas may be brought to the children objectively. A return to these objective demonstrations from time to time will be found an excellent aid to children in retaining the ideas they have once learned. The memory of children needs to be continually jogged.

151. Business Usage to be Followed with Fractions. Some time ago fractions with long terms were common in actual business and, as a consequence, deserved a place in the arithmetics of the day. These fractions have not been in use for some time now, but they are just beginning to be dropped from the arithmetic texts. At the present time, they have no place in the number work and may well be omitted altogether.

There is a common business notation of today that many texts are omitting which, however, should not be overlooked. As a half and the quarters are the most used business fractions they are commonly written without the denominator, which is 4 understood. Thus, $5\frac{1}{4}$ is 5^1, $3\frac{1}{2}$ is 3^2, $7\frac{3}{4}$ is 7^3.

152. Addition and Subtraction. Introduction to addition and subtraction of fractions should be through examples in which the same denominator is used, thus, $\frac{2}{8} + \frac{1}{8}$. Then, may come those examples where one fraction is reduced to its equivalent in which the denominator is the same as that of the other fraction, as, $\frac{2}{3} + \frac{1}{6}$, $\frac{2}{5} + \frac{3}{10}$.

In order to permit combinations by addition or subtraction fractions must be alike; i. e., must have the same denominators. A review of the fact that the denominator is the *naming part* of the fraction and the numerator the *numbering part* will be in place here.

Paper-folding will be found to be a great aid in introducing the changing of fractions to equivalent ones, with given denominators. The term, common denominator, may be introduced and its meaning worked out objectively at this time. When this is understood thoroughly the pupils should be encouraged to look for numbers with which to multiply both terms of the fractions,—different for each fraction, of course—so that the resulting fractions will all have the same denominator.

It is advisable not to refer to L. C. M. at this time. Emphasis is thus placed upon the idea of reduction of fractions to equivalent ones, and not upon the mechanical process of reducing to a common denominator through L. C. M. To carry out this plan it is necessary to use fractions with denominators which are easily factorable at first.

After the real meaning of changing fractions to those with a common denominator has been fully grasped, the

term L. C. M. and its use may be introduced. The following common form will be found easily comprehensible to the children.

2)	5	15	14	30.
5)	5	15	7	15.
3)	1	3	7	3.
	1	1	7	1.

Hence, L. C. M. $= 2 \times 5 \times 3 \times 7 = 210$.

As a rule, it is best to discourage contraction of notation since it often leads to confusion and inaccuracies. The symbolical language, which is one of the two things to be learned, must be learned correctly. Contraction of English is permissible in a telegram but telegrams are not the usual modes of speech. Care must be taken, therefore, not to permit such statements as "multiplying the fraction $\frac{2}{3}$ by 4 gives $\frac{8}{12}$."

The only variation encountered in subtraction from that in addition, is the case of mixed numbers in which there is no fraction in the minuend, or in which it is smaller than in the subtrahend. If the analogous case in integers has been well considered this will present no serious difficulty. It is a good plan to precede such examples with two or three similar examples in integers.

153. Multiplication. As in integers, multiplication should be taken up before all of the work in addition has been "completed." It is well to begin with the multiplication of a fraction by a whole number. Since the denominator only names the fraction, then $5 \times \frac{3}{16}$ means that 3 of a certain kind of number (sixteenths) is to be multiplied

by 5, and hence the result will be 15 of these sixteenths or, $\frac{15}{16}$, just as three cents or three oranges multiplied by 5 would give 15-cents or 15-oranges. When this fact has been thoroughly comprehended, it is well to accustom the pupils to be on the lookout for common factors between the integral multiplier and the denominator. If the pupils have been taught that the order of multiplying factors is immaterial, as was emphasized in Chapter XII, then they will have no difficulty in multiplying an integer by a fraction, as it reduces to that of multiplying a fraction by an integer.

Multiplication by unit-fractions is made a forerunner of the general case of multiplication of two fractions. For the first examples, select fractions in which the denominator of the unit-fraction is a factor of the other numerator, as, $\frac{1}{4} \times \frac{12}{19}$, which asks for $\frac{1}{4}$ of 12-nineteenths, just as $\frac{1}{4}$ of 12-cents or of 12-oranges might be asked for. The result will clearly be 3 of these nineteenths, as in the other two cases it would have been 3-cents or 3-oranges. $\frac{5}{4}$ of $\frac{12}{19}$ is only 5 times the value of $\frac{1}{4}$ or $\frac{12}{19}$. After the solution of several examples of this sort the pupils may be directed to multiply the numerators together and the denominators together for new numerators and denominators, if they do not discover this for themselves. The fact that a fraction is an indicated division should be brought out in connection with multiplication by unit-fractions. Thus, $12 \times \frac{1}{3} = \frac{12}{3} = 4$; also $\frac{1}{3}$ of $12 = 4$; and $12 \div 3 = 4$. Hence we may write $12 \div 3 = \frac{12}{3} = 4$. Later this fact is extended in division by a fraction to simpler complex fractions, as, $\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$. Sufficient work should be given in complex fractions to fix the idea that a fraction is an indicated division and to enable pupils later to clear up expressions of this nature which may arise from substitutions in formulas.

154. Mixed Numbers in Multiplication. Products of small mixed numbers, as $2\frac{1}{2} \times 3\frac{1}{3}$, can best be found by reducing each to an improper fraction. As a rule, however, it is preferable to perform the multiplication without making such a reduction. One illustration must suffice:

$6\frac{1}{4}$	$6\frac{1}{4}$
$8\frac{1}{3}$	$8\frac{1}{3}$
48	48
2	2
$2\frac{1}{12}$	$2\frac{1}{12}$
$52\frac{1}{12}$	$268\frac{1}{12}$

Only one error is liable to arise, that of placing the products of an integer and a fraction in the wrong column, as in the computation to the right. This error may be pointed out by asking the pupil who made the mistake to give the product of 7×9, which must give more than the required multiplication, but which is in reality much less than the product erroneously found. The figures found in the average multiplication of mixed numbers may not be as simple as in the example given but in many cases close approximations only are needed, as in the accompanying illustration.

$25\frac{2}{3}$
$12\frac{5}{6}$
50
25
5
21
326

155. Division. Division should be taught as the reverse of multiplication just as was done in the case of integers. It is well also to follow the order in multiplication, similarly to the fashion for integers.

The first examples given to the class should be division of a fraction by a whole number, the numerator of the fraction containing the divisor as a factor, as, $\frac{12}{19} \div 4$. This 12-nineteenths divided by 4 gives 3-nineteenths just as 12-cents, or 12-oranges divided by 4 would give 3-cents, or 3-oranges. If the numerator of the fraction is not a factor of the divisor it may easily be reduced to one which is, as, $\frac{4}{5} \div 3 = \frac{12}{15} \div 3 = \frac{4}{15}$, which the pupils will soon see, is equivalent to multiplying the denominator by the integral divisor.

Division of a whole number by a unit fraction is next in order, for example, $12 \div \frac{1}{4}$. The question really is, "How many fourths are there in 12?" Applied to the foot-rule which the pupils can handle objectively it becomes, "How many quarter-inches are there in a foot?" In order that the foot-rule may not appear to be a sort of magic wand, other objects should also be used. As an extension comes $12 \div \frac{3}{4}$. The question is, "How many groups of 3 one-fourths can be made from 12?" As there are 48 one-fourths in 12, there will be $48 \div 3$, or 16 such groups.

Finally, comes division of one fraction by another fraction as, $\frac{12}{17} \div \frac{2}{3}$. There are $\frac{36}{17}$ one-thirds in $\frac{12}{17}$, hence there are $\frac{36}{17} \div 2 = \frac{18}{17}$ of two-thirds in $\frac{12}{17}$. After solving several examples in this way, the pupils should be directed to invert the divisor and multiply if they have not already discovered this method of precedure for themselves.

156. Division by Comparison of Numerators. Again, in a division both fractions may be reduced to the same denominator; thus, $\frac{2}{3} \div \frac{4}{5}$ would become $\frac{10}{15} \div \frac{12}{15}$ which is

the same as 10 of a certain kind of unit divided by 12 of the same kind of unit; 12 cents divided into groups of 3 cents in each, or 12 oranges divided into groups of 3 oranges in each. The results may then be compared with those obtained through the multiplication by the divisor inverted. In most cases, the pupils if reasonably shrewd will see this from their own observation. The teacher can not be too rigorous in requiring all results to be verified, as was done with integers. As a rule, mixed numbers occurring in division would better be reduced to improper fractions.

157. All in One Step—Cancellation. As soon as the pupils are fully familiar with the fact that a fraction can be used to indicate a division, as $\frac{3}{4} = 3 \div 4$, they should be encouraged to express every operation in this manner, before performing the actual division. The advisability of such a plan has already been pointed out in 58. As most children enjoy cancellation they will take to it readily enough, provided that the teacher has prepared them for it by preliminary work in the solution of a sufficiently large number of cancellation examples.

158. Aliquot Parts. Some of the aliquot parts of 100 can very profitably be brought into use in multiplication and division. It is far better to consider only a few at first and add to these as time goes on. To make this work effective it is necessary that the teacher insists on the use throughout of every short cut which has been learned whenever opportunity arises, and also, that he employs from time to time, numbers admitting their use in dictated examples and problems. It is well to introduce aliquot parts through a problem in which the direct multiplication would be quite a long process, but which becomes merely a mental process by the use of aliquot parts, as in the example $1236 \times 16\frac{2}{3}$. The applied value of aliquot

parts is often lost because the teacher does not sufficiently emphasize its value in those instances in which a part of one number is a useful aliquot part, as, $33\frac{1}{3}$ in $533\frac{1}{3} \times 5451$, or 25 in 725×9348.

159. What Pupils Should Know. It is absolutely necessary that pupils who have completed the work in fractions have a clear understanding of the following:

1. What operations do not change the value of a fraction.
2. What operations multiply a fraction.
3. What operations divide a fraction.
4. A thorough comprehension of the four fundamental operations with fractions.
5. That all final results are to be placed in their simplest form, showing both what that form is and how it is obtained from the given fraction.

XIII. EXERCISES

1. Why should we expect children to have difficulties in the study of fractions?
2. Give a résumé of the various kinds of fractions in use before modern times.
3. How have the modern texts, until very recently, handled fractions? Discuss fully.
4. What knowledge of fractions should the children gain during the first three years? Illustrate how this knowledge may be acquired.
5. Analyze the difficulties the children are likely to experience in these first years.
6. What is meant by calling the numerator and denominator the "numbering" and the "naming" parts of a fraction? Illustrate.
7. Explain that fractions are units. Illustrate.
8. Show the use of paper-folding in presenting the idea of fractions to a class.
9. Show how pupils are liable to memorize a mechanical manipulation of symbols rather than the real idea. How may this be overcome?
10. How would you introduce to a class, the reduction of mixed numbers to improper fractions and *vice versa*.
11. How do business men write $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{4}$? Where would these arise?

12. Explain, briefly, how addition of fractions should be introduced to a class.

13. State the order in which addition and subtraction of fractions are to be taken up. Illustrate.

14. Suggest at least four suitable examples in each for a grade class.

15. It is very important that the children understand the meaning of changing fractions to common denominators. How would you go about giving them that understanding? Explain fully.

16. Arrange in order of magnitude: $\frac{15}{16}$, $\frac{47}{48}$, $\frac{31}{40}$, and $\frac{23}{30}$.

17. Define multiple and divisor. Give two multiples and two divisors of both 36 and 50.

18. Define L. C. M. and H. C. D. Find the L. C. M. and H. C. D. of 42, 35, 63, 70; of 28, 84, 21, 12.

19. Show that the product of the L. C. M. and H. C. D. of two numbers equals their product.

20. Perform the following subtractions by the Austrian method:

$6\frac{7}{8}$	$5\frac{3}{4}$	$14\frac{1}{3}$	$9\frac{4}{5}$	$8\frac{5}{12}$
$2\frac{3}{8}$	$2\frac{1}{2}$	$5\frac{2}{3}$	$3\frac{1}{4}$	$3\frac{5}{6}$

21. Discuss fully the use of L. C. M. in addition, and the proper time to introduce it.

22. Discuss the disadvantages of contracted notation in early work in fractions. Illustrate.

23. Explain the way you would introduce multiplication of fractions to a class. What use can be made in this connection of paper-folding and similar objects? Illustrate.

24. State the order in which the subject of multiplication should be taken up. Illustrate.

25. Find the following products:

$24\frac{3}{8}$	$45\frac{5}{8}$	$132\frac{1}{7}$
$12\frac{3}{8}$	$16\frac{5}{8}$	$42\frac{3}{8}$

26. Suggest six graded examples in the multiplication of mixed numbers for a grade class. When should the mixed numbers not be reduced to improper fractions? Why?

27. Explain how the division of fractions is the reverse of multiplication.

28. Explain the introduction to a grade class of division of fractions.

29. Outline the order of taking up this subject.

30. Explain fully the reason for inverting the divisor. How far should the pupils be required to understand this reason?

31. What part of his salary does a man save, who spends $\frac{1}{2}$ of his income for rent, $\frac{1}{8}$ for board, $\frac{1}{10}$ for clothes and $\frac{1}{4}$ for other expenses? How much will he save out of a salary of $1600 per year?

32. In a school of 40 pupils the number of boys equals $\frac{3}{5}$ the number of girls. What part of the total number are girls? boys? How many are boys? girls?

33. A grocer uses $15\frac{1}{2}$ ounces for a pound. What fraction of his sales by weight does he cheat his customers? What would be the loss to a customer buying 200 pounds of sugar at 6 cents per pound?

34. A man owning $\frac{3}{4}$ of a factory sells $\frac{2}{3}$ of his share for $21,000. What part of the factory does he still own? What part did he sell? What was the total valuation?

35. A man sold two horses for $120 each. On one he gained $\frac{1}{3}$ and on the other he lost $\frac{1}{3}$. Did he gain or lose and how much?

36. A manufacturer makes $\frac{1}{25}$, wholesaler $\frac{1}{5}$, and the retailer $\frac{1}{4}$ on an article sold at retail for $20. What was the cost of manufacture?

37. A man owning $\frac{5}{8}$ of a section of land sells $\frac{1}{4}$ of his land, and later sells $\frac{1}{3}$ of the remainder; how many acres has he left?

38. What number increased by $\frac{1}{10}$ of itself equals 372?

39. 533 equals $\frac{11}{10}$ of what number? Compare with problem 38.

40. Water expands $\frac{1}{10}$ in freezing. If a cubic foot of water weighs $62\frac{1}{2}$ pounds what does a cubic foot of ice weigh? How many cubic feet of ice weigh $62\frac{1}{2}$ pounds?

41. If $\frac{3}{4}$ of A's money equals $\frac{1}{2}$ of B's money and they have $40 together, how much has each?

42. A boy buys papers at the rate of 3 for 5 cents. The first one-half he sells for 3 cents each but receives only 2 cents each for the remainder. What part of the buying price did he gain or lose?

43. Simplify: $\dfrac{\frac{3}{5}+\frac{3}{4}-\frac{5}{8}}{\frac{1}{8}+\frac{1}{7}}$.

44. Simplify: $\dfrac{6\frac{2}{3}}{3\frac{1}{3}} \div \dfrac{7\frac{1}{3}}{3\frac{1}{3}}$.

45. Criticize the following:

i. $\dfrac{4-3+3}{5-2+3} = \dfrac{1}{3}$. ii. $\dfrac{6}{6} = 0$. iii. $\dfrac{8-5+9}{9+12-5} = \dfrac{8}{12} = \dfrac{2}{3}$.

46. Suggest at least two problems in fractions which you would expect your pupils to find difficult.

47. Explain the advantages of stating all the operations in a problem before carrying out any of them.

48. Give a complete symbolical statement of problems 33, 34, 36, and 37 before attempting any of the operations. Compare this form of solution with one in which each operation is worked out with the progress of the reasoning.

49. Suggest at least three problems in which one may take advantage of this principle.

50. What are aliquot parts? Illustrate. Explain their use in multiplication and division by several illustrations.

51. Perform the following operations in the simplest manner:

i. $233\frac{1}{3} \times 432$;
ii. $31200 \div 33\frac{1}{3}$;
iii. $4000 \div 125$;
iv. 325×412;
v. $716\frac{2}{3} \times 726$;
vi. $450 \div 50$;
vii. 25000×432.

52. Suggest six examples on aliquot parts suitable for grade work.

53. Show that aliquot parts are based upon the principle of reciprocal operations.

54. What should the pupils know after they have "completed" the work in fractions? Illustrate each.

55. What kind of fractions should be avoided? Illustrate.

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CHAPTER XIV

WEIGHTS AND MEASURES

160. History. Only the very briefest sketch can here be attempted of this most interesting phase of the development of arithmetic. The teacher will be well repaid by going more extensively into the subject and by interesting his pupils in reading and reporting to the class on various phases. These reports by the pupils can be introduced in connection with either the geography or arithmetic lessons. Wherever used they will add interest and life to the work. For this purpose the mathematical histories listed at end of chapters, of which at least one is recommended for every school library, and the standard encyclopedias, will be found useful.

From the earliest time of which there are any records, some forms of weights and measures have been used. The names given to the units of these weights and measures have usually been taken from the names of grains or of some part of the human body. Many of the names are still retained, although they have undergone a series of changes as to meaning. Thus, the *carat* bean was used for determining the unit of weight by the Hindus, as was the wheat *grain* by the Egyptians, Romans, English, and most of the later nations. In England the statute pound contained 7680 grains, and the Troy pound, 5760 grains. As a result of this variation, some of the old arithmetics stated that "32 grains of wheat make 24 artificial grains."

Avoirdupois pound, called in early English *haverdupois*, meaning "goods of weight," was introduced into England

from France about 1300 and contained 7680 grains as did the old statute pound, which is now 7000 grains. At first this avoirdupois pound was used for weighing meats only; later, however, it was used to weigh all heavy articles.

The English *stone,* 14 pounds, is interesting from the fact that it is the origin of the long ton found in schoolbooks of a few decades ago, but is now used only in the customhouse and in mines. Eight *stones,* or 112 pounds, was the integral number of *stones* which gave a weight of at least 100 pounds and was, hence, called a hundredweight. The ton of 20 hundredweight, therefore, contained 2240 pounds.

Pennyweight has its origin in the weight of a penny.

The *cubit* of the early Egyptians, Babylonians, and Hebrews was the length of the fore arm, from the elbow to the tip of the middle finger. This unit is considerably older than the foot, which is in almost universal use, and needs no explanation. The *inch* has been variously defined: by the English it was held to be the length of three rounded barleycorns, by the French and the Scandinavians it was the length of the thumb from the first joint to the tip of the nail, and they give that name to the inch, by the Romans it was held to be one-twelfth of a foot. The Romans also designated the *ounce* as one-twelfth of a pound and etymologists have traced the English terms, "inch," and "ounce," both to the Roman *unciea.* Henry I decreed that the length of his arm should be the legal length of the *ell,* the *yard* of the present day. Thus some people, today, measure the length of the yard from the chin, or nose, to the end of the outstretched arm.

Other units of length have been the *hand,* meaning the width of a hand; the *span,* the length from the tip of the thumb to the end of the little finger when the hand is outstretched; *finger-breadth; fathom,* the length of the outstretched arms; *pace,* refering to the length of a step;

hair's-breadth, used to denote a minute space. Other interesting units of length are the *rod,* the length of some measuring rod; *furlong,* a furrow long; *mile (mille passuum),* a thousand paces; *nautical* or *geographical mile (knot),* the length of 1′ of arc of a great circle on the earth's surface; GUNTER'S *chain,* 66 feet.

There is no accurate information regarding the derivation of the *gallon.* The *quart* is, of course, one quarter of the gallon. *Gill* likely comes from the Latin, *gilla,* a drinking glass. *Bushel* is supposed to have meant a certain-sized box. *Peck* is believed to have come from pack, meaning, perhaps, a load to carry. *Acre* meant merely an open piece of ground but is supposed to have been the amount of land a pair of oxen could plow in one day. For most of the other units there are less interesting connections, or none at all.

161. Choice of Units by Early Peoples. The units selected by the early peoples, as by those of more modern time, have always been of convenient size for service. Selections were sometimes made to suit some special purpose. For instance, instead of changing prices, merchants of England used different units of weight in buying and in selling. Land was divided into 40 *perches* to the *acre,* in order to connect the wages paid for work with the amount of work on land which could be accomplished in a day. Yarn was divided into *sacks* and *tods,* depending upon the average amount of work which a spinner could accomplish per day. In our day of modern inventions, such selections of units are, of course, out of the question.

In consideration of the variety of units that have been mentioned, any approach to uniformity among the early nations was obviously out of the question. Regulations of some of the units as an effort toward establishing a degree of uniformity were made as early as WILLIAM THE

CONQUEROR, and have been made from time to time thereafter, but in the main without satisfactory results. Some of this was due to such trivial facts as that the merchants would rather change weights than change prices. To complicate matters, these changes varied with the community and they were even different for the days of the week in the same community.

About the beginning of the eighteenth century, England defined the Winchester *gallon* as 231 cu. in. and the Winchester *bushel* as 2150.42 cu. in. These measures have been standard in the United States since 1834, but were replaced in England by the imperial gallon and bushel containing 277.274 cu. in. and 2218.19 cu. in. in 1824. In 1836 the Secretary of State had copies of the then accepted yard sent to the various states of the Union. The present English unit of length, the yard, was adopted in 1855, a copy of which was presented to the United States. This was found to be $\frac{1}{1000}$ of an inch shorter than that of the formerly accepted unit. By an act of 1893, the yard was defined as $\frac{3600}{3937}$ of the meter.

162. Standardization of Weights and Measures. Congress has enacted laws governing the weights and measures and their standardization from time to time. The United States National Bureau of Standards was established in 1901, whose scope has since been greatly modified. It is, therefore, well within the last century that the standardization of the weights and measures has been attained, although even yet no international units have been decided upon.

163. Monetary Standards. The fascinating story of the development of moneys can hardly be touched upon here. It began with direct barter that led up to a medium of universal value, such as cattle in Rome, fish in some of the north countries, or wampum among the Indians, and finally,

precious metals became a medium. Many a schoolboy has wondered why the English could not have found a different name for their units of weight and of money than *pound*. This double use of the word is not due to a lack of words, but to the fact that in early times, money was reckoned not in *tale* (the stamp upon the coin) but in weight. This was the usual custom among all peoples, reference already having been made to the Roman *as*. The coins do not today possess these original values, as the result of their having been debased by the sovereigns of the different countries.

The east German traders, called Easterlings, brought to England a coin which was given the name Easterling and later modified to Sterling the name now in use. "Farthing" comes from the *fourthing* of a penny. The use of English money prevailed until a late day with the early colonists. Each separate colony, also, had its own coins. This gave rise to a great deal of matter in the arithmetics on exchange, a subject which has received undue importance until very recently. The origin of the dollar mark is often taken to be a combination of U and S but Cajori has very clearly pointed out, in an article in the Dec. 1912 issue of the Popular Science Monthly, that its origin is that of the Spanish mill dollar.

164. First Four Years. All of the work in denominate numbers for the first four years should be informal. A sufficient amount of number work, however, should be given during these years involving the common measures, such as foot, inch, yard, quart, pint, gallon, pound, ounce, and ton, to enable any child who is compelled to drop out of school by the end of this period, to have gained a comprehension of these units and their use in measurements.

It is as easy and as interesting to add inches or quarts as to add marbles or tops. Reductions should be made

only from one denomination to the next ascending or descending as, the reduction from quarts to gallons, or pints, but never pints to gallons, or gallons to pints. It is a good thing to give much practice at this time in the actual handling of the measures; for example, finding lengths in feet and inches at first, and later longer lengths in yards; or, showing by the use of sand or water that there are 2 pints in a quart, 4 quarts in a gallon, and 8 quarts in a peck. In linear measurements no account should be taken of fractional parts. So that a length is called a little over 5 feet or a little over 9 inches, and so on.

The number of ounces in the pound and the number of pounds in the ton might better be learned directly, without the process of weighing, except that the children must be acquainted with the process of weighing and the difference between the bulk and weight of an object. Weighing the same capacity of different substances will help greatly in teaching them to distinguish between weight and capacity. An important subject for this informal work is the divisions of time, such as day, week, month, year, hour, minute, and second, including the telling of time by the clock.

Simple problems in addition and subtraction can be very easily made up from any of the measurements but judgment is necessary in the selection of reductions suitable for examples in multiplication and division. For instance, during the study of multiplication by 2 the reduction is of pints to quarts and its reciprocal in division; by 3, of feet to yards and the reciprocal; by 4, of quarts to gallons and of pecks to bushels and their reciprocals; by 8, of quarts to pecks and their reciprocals. With a little practice the teacher will be able to suggest any number of problems based upon these relations. The relation of the inch to the foot furnishes a simple introduction to the elements of fractions as previously mentioned.

Square measure arises directly from linear measure first in the drawing of squares and later in drawing rectangles. Measuring areas of desks, floors, books, and so on, furnishes a great variety of usable material.

Much of the early number work in measures has already been gone into in a previous lesson and needs no further elaboration here.

165. Fundamental Ideas. As an introduction to the formal study of denominate numbers a short review of the informal work just suggested is recommended, in which the teacher pays special attention to any pupils who may not have had this preliminary work. Let the fundamental object be to give the children a clear conception of the various units and their practical use in everyday life.

Teach the art of measuring. Many a child who is perfectly familiar with the tables and their application in the most complicated reductions often has only the slightest idea, at most, of the meaning of the various elements which arise, as, for instance, the size of a gallon or a bushel, the extent of an acre, or the length of 100 feet or 100 yards. It is a good plan, therefore, to pay chief attention to the measurements used in the community. No measurement must be neglected since the teacher has no way of knowing how or where the future life of the children may be spent.

Distances frequently referred to, of which the average person has little idea, are 50, 100, and 300 feet, 220, 440, and 880 yards. Many of these will appeal especially to the boys, as they often arise in athletic events, as do also 60 and 90 feet in the baseball diamond. These distances can either be "stepped off" or measured by a heavy cord that does not stretch and that is knotted every 5 feet. Other measurements which may be ascertained in this manner are width of the sidewalk, street, and lot, length of a city block, dimensions of the school building, and so on.

166. Indirect Measurements. As most of our measurements, excepting time, length, and weight, are derived indirectly, that phase of the subject must receive considerable attention. All areas are found from two linear measurements. The fundamental idea in finding the area of a rectangle must be thoroughly studied; namely, that the rectangle is divided into as many strips as the rectangle is linear units in one direction, that each strip contains as many units of area as the number of linear units in the other dimension and that, hence, the total area must be the area of one strip multiplied by the number of strips. See **62.** Energy is often wasted by giving undue importance to the fact that linear units can not be multiplied by linear units to get units of area.

As soon as the child is fully acquainted with this fundamental idea of area so that he will not puzzle over "reducing square feet to cubic feet" it would seem safe to permit him to say that he gets area by multiplying length by breadth. It is only a contraction of the above, in which the child means the multiplication of the *number* of units of length by the *number* of units of breadth. This contraction is accepted readily enough in Europe and is being accepted more and more in the United States.

A most common indirect measurement is that of finding the contents in bushels of a load of grain by weighing. Much attention should be given to the solution of problems of which the following is typical:

What is the value of a load of oats at 34 cents per bushel for which the gross weight is 4640 and the tare 2780 pounds, respectively?

It is well to point out that large weights like these are weighed only to within 10 pounds. The standards used in your state for the various grains should be employed in all such problems. Lack of uniformity in the various states

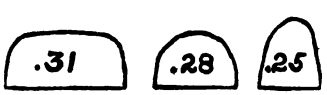
may be pointed out in the 7th or 8th grades when reviewing this topic in connection with mensuration. Many of the best problems in mensuration involve a knowledge of denominate numbers which should hence be given a thorough review in connection with mensuration.

167. Isolated Units that Should Be Learned. Troy and apothecaries' tables are now omitted—and that rightly—from most arithmetics. It may be well to mention that all weighing in a drug store is avoirdupois, except in prescriptions where the metric is fast replacing the apothecaries' system. There are several isolated units which should, however, be learned:

1. A pint equals 16 fluid ounces.
2. One cubic foot contains about 7½ gallons.
3. It is estimated that 1⅘ cu. ft. make a heaped bushel. A bushel of corn is supposed to be contained in 2½ cu. ft. of ear corn or in 3½ cu. ft. of unhusked corn.
4. Thirty pounds of ensilage is estimated to be one cubic foot and is considered a ration.
5. A cubic foot of water weighs about 62.5 pounds.
6. A cubic yard of dirt is called a load.
7. Varying formulas are used in different communities for estimating the number of tons of hay in a stack. Some of the more common are:

$$L \times W \times \tfrac{1}{3} O = \text{number of cubic feet.}$$
$$[W + O]^2 \div 16 \times L = \text{number of cubic feet.}$$
$$F \times O \times W \times L = \text{number of cubic feet.}$$
$$[R \div 4]^2 \times E = \text{number of cubic feet.}$$

L means the length of the stack, W the width, O the distance over, R the distance around, E estimated height, and F is a fraction varying with the shape of the stack as shown in the accompanying figures.



The number of cubic feet taken to the ton depends upon the age of the hay as follows:

512 cu. ft. = 1 ton new stack.
422 cu. ft. = 1 ton stack 5 months old.
342 cu. ft. = 1 ton stack 1 year old.
216 cu. ft. = 1 ton stack 3 or 4 years old.

8. One measuring cup equals one-half pint.
A measuring cupful of flour weighs one-half pound.
A measuring cupful of corn meal weighs one-third pound.
A measuring cupful of butter weighs one-half pound.
A measuring cupful of chopped meat weighs one-half pound.
A measuring cupful of granulated sugar weighs one-half pound.
A measuring cupful of liquid weighs one-half pound.

9. A cord of wood is a pile 4 by 4 by 8 feet.

10. A board foot is a board with a surface of 1 sq. ft. and a thickness of 1 in. or less. Lumber is generally sold by the 1000 board feet stated in M. A bundle of shingles contains 250 shingles averaging 4 in. wide.

11. Measurement of city lots in front feet or square feet and farm lands in sections and fractions thereof must be entered into thoroughly. If advisable the more extended study may be delayed until the last years.

168. Reductions. Much more time was formerly given than at present to reductions of denominate numbers and the problems were much more complex. Reductions involving more than three denominations arise very infrequently and, hence, their omission is advised. The following order is suggested:

i. Descending of two denominations, as 3 qt.—1 pt. to pt.
ii. " " " " " 3 gal.—1 qt. to pt.
iii. " " three " " 4 gal.—3 qt.—1 pt. to pt.
iv. " " two " " 3 gal.—1 pt. to pt.
v. Ascending " " " " 10 qt. to gal.—qt.
vi. " " three " " 13 pt. to gal.—qt.—pt.

Some schematic form which is clear and takes little time, is advisable. The following are suggestive:

Descending	Ascending
4 gal.—3 qt.—1 pt.	2) 13 pt.
4	4) 6 qt.—1 pt.
19 qt.	1 gal.—2 qt.—1 pt.
2	
39 pt.	

169. Four Fundamental Operations. It is not necessary to delay the four fundamental operations of denominate numbers until all of the reductions have been learned, but care must be exercised to see that no reduction not understood is needed in the operation to be performed. Require all final results to be placed in the simplest form.

1. Addition:

Use the common form given below.

3 gal.	—3 qt.	—1 pt.	
4	2		
2	1	1	
9	6	2	(This auxiliary line may be used at first
10	3		but is to be discontinued as soon as possible.)

2. Subtraction:

Subtraction should be taken up shortly after addition and carried on simultaneously with it. It is a good plan to introduce subtraction with a few examples such as those given below in which no borrowing is required. By the Austrian method example (1) below becomes: 1 pt. and 1 pt. equals 2 pt.; 1 qt. and 2 qt. equals 3 qt.; 2 gal. and 3 gal. equals 5 gal. Example (2) becomes: 3 qt. and 3 qt. equals 2 qt. and 1 gal.; 1 gal., 2 gal., and 3 gal. make 6 gal. This requires a little more insight into reduction of denominate numbers than the other form of borrowing but is more practicable when once understood. To bar the Austrian method because it requires a better knowledge of

reduction is like barring multiplication because we can do the work by addition without needing to learn the multiplication tables.

(1)

5 gal.	3 qt.	2 pt.
3	2	1
2	1	1

(2)

6 gal.	2 qt.
2	3
3	3

3. Multiplication:

As in addition, the pupils may be permitted to give preliminary results at first. This practice, however, is to be discontinued as soon as possible.

2 gal.	3 qt.	1 pt.
		5
10	15	5
14	1	1

4. Division:

Division, by partition and by measurement, must be understood thoroughly by the pupils as two separate problems. Mention of these terms—partition and measurement—would better be omitted, but the idea involved must be grasped. Care must be taken, also, that all processes of reduction and borrowing be perfectly clear. The following is suggested:

1, examples on partition requiring no borrowing;

2, introduction of those requiring borrowing;

3, division by measurement.

As a rule, both dividend and divisor would better be reduced to the same denomination in examples like the last.

(1)

$$3)\ \underset{2}{6\text{ gal.}}—\underset{1}{3\text{ qt.}}$$

(2)

$$\begin{array}{r} \\ 5)\ 13\text{ gal.}—3\text{ qt.} \\ \hline 2\quad 3 \end{array}\qquad 15$$

(3)

$$(4\text{ gal.}—2\text{ qt.}) \div (1\text{ gal.}—1\text{ qt.}) = 18\text{ qt.} \div 5\text{ qt.} = \tfrac{18}{5} = 3\tfrac{3}{5}.$$

Many excellent problems arise under division, especially under division by measurement, and others will arise later under mensuration.

XIV. EXERCISES

1. Give a historical report of the development of the following:

i. Weights.
ii. Linear Measure.
iii. Land Measure.
iv. Dry Measures.
v. Variation of Dry Measures in the United States.
vi. Liquid Measure.
vii. Compare Liquid and Dry Measure.
viii. Money of Foreign Countries.
ix. Money of the United States.
x. Laws and Regulations.

2. Explain fully how you would use this work in the grades.

3. What should each child know of denominate numbers at the end of the fourth year? Upon what should the emphasis be placed during these early years?

4. Explain, as you would to an elementary class, some one unit, such as the inch.

5. What are the main reasons for taking up denominate numbers this early?

6. What should be the main object of the study of denominate numbers?

7. What is the approximate length of the library, of the main building, of a city block, of the width of a street?

8. What part of an acre is the average city lot?

9. Give the dimensions of the baseball diamond.

10. Give the following in feet:

i. A horse 15 hands high.
ii. A march of 2500 paces.
iii. A lake 4 fathoms deep.

Give the following in miles:

i. A ship's speed of 12 knots per hour.
ii. A race course 6 furlongs long.
iii. A flight of 5 leagues.

11. Compare a speed of 30 miles per hour with 40 feet per second; 400 yards per minute with 22 feet per second; 25 miles per hour with 700 yards per minute.

12. What are indirect measurements? Give three illustrations.

13. Explain as you would to a class the introduction of square measure.

14. Criticize "12 ft. times 6 ft. gives 72 sq. ft."

15. Answer the following question put by a pupil: "How are square feet reduced to cubic feet?" What is the trouble here? How may it be remedied?

16. What fractional part of an acre is a garden 40 by 60 feet? 16 by 30 feet?

17. How many acres are there in a pasture $\frac{1}{16}$ by $\frac{1}{8}$ mile? 20 by 60 rods?

18. Explain the meaning and use of township and range lines.

19. Locate, by means of a drawing, the following:

i. S $\frac{1}{2}$ of N-W $\frac{1}{4}$ of sec. 4—23—46.
ii. N-E $\frac{1}{4}$ of N-W $\frac{1}{4}$ of sec. 6—19—25.

How many acres are there in each?

20. What is meant by the expressions "gross," "tare" and "net" weight? Illustrate.

21. How should a class be led to see the process in finding the number of bushels of grain in a load?

22. Suggest at least three problems involving the terms in Ex. 20 that are suitable for a grade class.

23. How many bushels and what weight would be used to plant 40 acres of corn, clover, wheat, and oats?

24. What is the cost at 35 cents per bushel of a load of ear corn, sold Oct. 15, if the gross weight is 4620 lb. and the tare 2740 lb.?

25. State three problems suitable for a grade class involving the measuring cup.

26. Give an explanation, suitable for a grade class, of cubic measure.

27. Give an explanation, suitable for a grade class, of the difference between weight and capacity.

28. How many bushels of ear corn will a crib 12 by 32 by 10 feet hold?

29. About how many cubic feet of water does a barrel hold?

30. Discuss the order of reductions suggested under 168.

31. State six simple examples in reduction suitable for oral work.

32. Explain as to a grade class some example in reduction.

33. State and explain each of the four fundamental operations as applied to denominate numbers.

34. Suggest an example illustrating each operation which would be suitable for a grade class.

35. Distinguish fully between the two classes of division. Illustrate.

36. Suggest two examples suitable for a grade class under each case in ex. 35.

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CHAPTER XV

DECIMAL FRACTIONS

170. History. Decimal fractions are an extension of the Hindu number-system to fractions. The authorship of the decimal system is in doubt although it is generally credited to SIMON STEVIN, a Flemish mathematician, who in 1584 published a set of interest tables in which he actually used our present idea of decimals. About fifty years before this, CHRISTOFF RUDOLFF, a German mathematician, divided by powers of 10 by cutting off with a comma as many figures from the right of the number as there were zeros in the divisor. For the use of this fundamental idea of decimals their invention is also ascribed to Rudolff.

It is almost incredible that over 1000 years should have elapsed before the simple idea of the Hindu number-system was extended to include fractions. Not only this, but the idea of decimals was staring the mathematicians and computers of southern Europe in the face when, about 450 years before their invention, they extracted square root by annexing ciphers and dividing the result by 10 raised to a power equal to half that of the number of ciphers annexed. The Hindus had used this method of extracting the square root much earlier.

Decimals did not come into actual use, however, until about 200 years after their invention and even today do not receive recognition commensurate with their beautiful simplicity. And the United States has for a number of years been stubbornly opposing the adoption of the metric system

of weights and measures which is the final step in our number-system.

Decimal notation has undergone several changes. Stevin wrote 42⓪7 ①3② or $4^{0}2^{1}7^{2}3$ for 42.73. A little later came $42^{0}\cdot7^{I}\cdot3^{II}$ according to which .00006 would have been written 6^{V}. Another form which prevailed until the eighteenth century in some books, was, 42 |73. The point came into use before the middle of the seventeenth century and has finally been universally adopted except in Germany, where the comma was introduced for the same purpose at about the same time and is still used. In England, the point is placed higher than in the United States, thus, 42·73.

It is unfortunate that the notation selected was not one which distinguishes the unit's place from the other figures of the number, as 32574 in the place of 325.74, because the 2 and 7 are 10 and $\frac{1}{10}$ times their value in unit's place, 3 and 4 are 100 and $\frac{1}{100}$ their value in unit's place, and so on. For every figure in 10's, 100's, and so on, there is a corresponding $\frac{1}{10}$th, $\frac{1}{100}$th, and so on, but none for the units, a fact which such a notation as the above would indicate. Stevin's notation, with the zero above the unit's figure, possessed the potentiality of such a system as that herein suggested. Nevertheless the advantages of the decimal notation over common fractions are very many and the teacher is urged to emphasize the use of this simple extension of our number-system.

171. Informal Introduction. The child may be informally introduced to decimals through United States money. He first learns the use of the point in separating dollars from cents which he employs in writing expressions involving dollars and cents. As soon as he is accustomed to handling numbers of three digits he may be given simple examples covering the four fundamental processes. At first

these examples should require no "carrying over the decimal point." They may be such simple problems as,

"If John earned $1.35 in January and $2.15 in February, how much did he earn during the two months?"
"What will 6 chairs cost, if $2.15 is paid for each chair?"
"What is the cost of one chair, if $8.40 is paid for 4 chairs?"

These may be followed by problems involving cents only, as,

"What is the cost of 6 books at 15 cents each? at $.15 each?"

After these come problems involving the use of the zero, as,

"What is the cost of 6 pencils at 5 cents each? at $.05 each?"
"What is the cost of 6 bushels of wheat at $1.05 per bushel?"

During this early stage the wise teacher will arrange for problems that require carrying to the left or to the right of the decimal point before introducing those that require carrying over it. Thus, "What will 6 chairs cost if $2.35 is paid for one chair?" Much handling of United States money problems in this incidental manner will be found very helpful later in that it presents a new subject through material with which the children are already familiar. Along with this, although not going into the theory to any extent, the teacher would do well to bring out the relations between the number to the left and to the right of the decimal point; namely, that it takes ten of the latter to make one of the former. The name, "decimal point," may be mentioned for the first time during this incidental introduction.

172. Formal Introduction. Decimals may be introduced from two quite different points of view: 1, as a new notation in the writing of fractions; 2, as an extension of the

Hindu number-system. As each has its own advantages, both will here be presented in order that the teacher may select the one best suited to the text used and to his own inclination. The pupils should not be confused, however, by the use of both.

173. First Method. Considered as a new way of writing fractions, decimals are merely a new notation; a new way of expressing old ideas. In \$2.45 there are 2 and $\frac{45}{100}$ dollars. Let the teacher first suggest such fractions as $\frac{3}{4}$, $\frac{2}{5}$, etc., for reduction to fractions with denominators which are powers of 10. He may now explain that there is a more convenient way of writing such fractions with a point, thus, .75, .4, etc.

Considerable practice should be given here in changing common fractions to this new notation by first changing their denominators to powers of 10. Care must, of course, be exercised to select only such fractions as permit of these reductions. Reduction of fractions in general to decimals would better be omitted in the early work—see under **179.** The reverse process, that of reducing decimals to common fractions, which is really the easier process, should be carried on side by side with the direct process.

174. Second Method. The introduction of decimals as an extension of the Hindu number-system will be more difficult for the pupils to comprehend than the one just given, but it is more direct and for that reason more effective. The unfortunate decimal notation, as was pointed out in the historical resume, makes the teaching of this new fractional language much more difficult than it would otherwise be. What the pupils must be made to understand from the start is, that digits increase 10-fold in value as moved to the left while they decrease at the same rate when moved to the right, and that digits to the right of the decimal point represent only *parts of units,* but increase and decrease

when moved to the right or left just as integers do. Thus .1 is $\frac{1}{10}$ of 1, .01 is $\frac{1}{10}$ of .1, etc.

United States money affords the only satisfactory illustration for this method of introduction. The pupils will readily see that in $.45 there are 4 dimes, each of which is $\frac{1}{10}$ of a dollar, and 5 cents, each of which is $\frac{1}{10}$ of a dime and $\frac{1}{100}$ of a dollar. [See under **18.**]

175. Remarks on Both Methods. Whatever plan is followed, the extension of the fundamental principle of place value to fractions in decimals must be made clear to the children and be firmly fixed in their minds. Much confusion and poor work in decimals are occasioned by the fact that the pupils do not understand the value of each decimal place; hence this must receive special attention and much drill. A corollary of this, namely, that annexing ciphers to the right of a decimal in no way alters its value, must also be clearly recognized by the pupils. An effective way of clearing up this matter is to show that annexing ciphers to a decimal is equivalent to multiplying both terms of a common fraction by powers of 10.

Confusion may also be avoided by permitting the use of the word *and* in reading numbers *only* in order to join the integral to the decimal part of the number. Thus, 254.65 should never be read, two hundred *and* fifty-four and sixty-five hundredths, but always, two hundred fifty-four *and* sixty-five hundredths.

At some time in the last two grades the children should be taught to read decimals as a business or scientific man reads them. For instance, 24.6532 would be read twenty-four, point, six, five, three, two; or twenty-four, point, sixty-five, thirty-two. This method of reading, however, should not be introduced during the early years while the children are busy learning the idea of place value and the names of the various decimal places.

176. Addition and Subtraction. Addition and subtraction of decimals need present no difficulties. Combinations in which there is no carrying should be used first; as, $.16 + .43$. It is a simple matter to make the children see that the sum of these decimals is 5-tenths and 9-hundredths. This fixes the position of the decimal point. Follow this with a few sums in tenths which add up to units, as $.5 + .2 + .3$, $.9 + .6 + .5$, etc.; then of hundredths which add up to even tenths, as $.07 + .08 + .05$, etc. Practice in such examples will show the pupils that the decimal point in the result always comes under the decimal point in the numbers to be added. Subtraction will give no serious difficulty if presented as the reverse of addition, just as outlined in integers.

177. Multiplication. It will be wise to begin multiplication with such examples as $\$65.23 \times 3$, in which there will be no carrying with figures to the right of the decimal point. These may be followed by examples in which there is carrying to either side of the decimal point but not over it, as 18.146×3, including some with zeros as 64.305×2, or 14.06×7, etc. Carrying over the decimal point should proceed by easy stages from examples such as $.5 \times 2$, $.4 \times 5$, etc., which give units; to $.6 \times 3$, $.7 \times 5$, etc., which give units and tenths; and finally, to the general case, 3.426×7.

An understanding as to the location of the decimal point is arrived at in a general way, from the fact that the product must be hundredths or thousandths if the multiplicand is hundredths or thousandths and the multiplier integral, just as the product of $\$58.32 \times 8$, or $\$62.47 \times 152$ must each contain cents. After much drill in multiplication by integers the process may be reversed, and integers be multiplied by decimals, as, $4 \times .3$, $8 \times .05$, etc. Inasmuch as the children know that the product will be the same whatever the order of multiplication, these problems may be changed to the

multiplication of decimals by integers. The law of the position of the decimal point can be arrived at inductively through the process of reducing each of two decimals multiplied together to common fractions. After the common fractions have been multiplied together, that product is re-reduced to a decimal. Comparison of the number of decimal places in the product and in the two factors will soon establish the law sought.

178. Division. Division may be introduced in a manner similar to that employed in multiplication; namely, by examples in which there is no "borrowing over the decimal point," as, $92.64 \div 2$, $60.75 \div 5$, $36.12 \div 3$, etc. After this comes division of a decimal by an integer requiring "borrowing over the decimal point," which leads up gradually to the most general case, as, $4.2 \div 3$, $5.6 \div 4$, $38.4 \div 12$, $32.5 \div 25$, $3.25 \div 5$, $3.25 \div 25$, etc. If division is treated as the reverse of multiplication and the results obtained verified by the multiplication of the quotient by the divisor, little trouble should be experienced in placing the decimal point after division. Examples giving remainders should not be attempted before the pupils are thoroughly familiar with placing the decimal point in the result.

Division of a decimal by another decimal is the final step. The law governing the position of the decimal point in the result for this most general case may be arrived at inductively by requiring the pupils at first to reduce all such cases to others in which the divisor is an integer; this to be accomplished by the multiplication of both divisor and dividend by the necessary power of 10. In fact, it is a question whether this mode of procedure is not a good one to follow in all cases of division of decimals.

A thorough drill in the multiplication and division of integers by powers of 10 should be included at this time. Because this work is so simple it is on no account to be

neglected, since it is of utmost importance to all future work in arithmetic. For this reason, the teacher is strongly urged to give much drill in this as well as in the more difficult forms of decimals.

179. Reductions. Except for those common fractions which can readily be changed to others with a denominator of some power of 10, the reduction of common fractions to decimals would better be postponed until after division of decimals. The reduction of $\frac{1}{18}$ to a decimal then becomes merely the division of 14.0000 by 18. Pupils inquiring about the reduction of a fraction, as $\frac{2}{3}$, to a decimal, before this time, may be told that such reductions will be considered in the future and their attention called to the matter when it comes up later for consideration. Such a delay will in no manner affect the work; it will exclude no work in the four fundamental operations with decimals or in the reduction of any decimal to a common fraction; while a sufficiently large number of fractions may be selected for reduction to decimals whose denominators can easily be expressed as powers of 10.

In order to establish the relation between common and decimal fractions some of the simple problems previously solved under common fractions may be translated to the decimal notation and solved as such. This may act as a forerunner of a more extended study of the three forms of fractions to be taken up later in connection with percentage, as will be discussed under **193.**

180. Circulating Decimals. The general reduction of fractions hence becomes a part of division of decimals. Circulating decimals would better be introduced through such forms as $\frac{2}{3}$, $\frac{5}{6}$, $\frac{5}{66}$, etc., which produce simple repetends. Approach to fractions producing long repetends as $\frac{3}{17}$, $\frac{3}{14}$, etc., should be made gradually.

In case the text does not give the language of circulating

decimals, the teacher would better do so. For example, $\frac{3}{14}$, which reduces to .214285171428517 . . ., is called a mixed circulating decimal to distinguish it from the pure circulating decimals such as $\frac{2}{3}$, or .666 . . ., in which all of the figures repeat. The former is commonly written $.2\dot{1}428517\dot{7}$, the dot being placed over the first and last figure of the repetend. $\frac{2}{3}$ would thus be expressed $.\dot{6}$, $\frac{5}{6}$ by $.8\dot{3}$, etc.

181. Decimals to Common Fractions. Reduction of circulating decimals to common fractions is fairly difficult and depends upon the summation of an infinite geometric series. For instance, the reduction of $.\dot{6}$ to a common fraction means to find the sum of
$\frac{6}{10} + \frac{6}{100} + \frac{6}{1000} +$ etc., for an infinite number of terms. The sum of such a series is found to be

$$S = \frac{a}{1 - r},$$

in which a is the first term (here $\frac{6}{10}$), r is the ratio between any term and the following one (here $\frac{1}{10}$). Thus

$$S = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{10} \div \frac{9}{10} = \frac{2}{3}.$$

In the case of $.8\dot{3}$ the common fraction equals

$$\frac{8}{10} + \frac{3}{100} + \frac{3}{1000} + \text{etc.}$$

The series $\frac{3}{100} + \frac{3}{1000} +$ etc., is summed as the previous one to which $\frac{8}{10}$ is added. This work is too difficult for children of the age at which decimals are usually considered, and hence would better be omitted except for the following arbitrary law depending upon the above mentioned series which may be given to the children if the teacher thinks it wise.

The repetend is written as the numerator of a fraction whose denominator contains as many 9's as there are figures

in the numerator with as many ciphers annexed as will be necessary to make the number of figures in the denominator equal to the number of decimal places in the first part of the repetend. For instance, $.\dot{6}\dot{3}$ would become $\frac{63}{99}$, or $\frac{7}{11}$; $.0\dot{6}\dot{3}$ would become $\frac{63}{990}$, or $\frac{7}{110}$; $.00\dot{6}\dot{3}$ would become $\frac{63}{9900}$, or $\frac{7}{1100}$, etc. Again, $.5\dot{6}\dot{3}$ would become

$$\frac{5}{10} + \frac{63}{990} = \frac{5}{10} + \frac{7}{110} = \frac{62}{110} = \frac{31}{55}.$$

182. Approximations. We have already considered one form of approximations; namely, computations with integers. We shall now look at another phase of approximations; namely, that of omitting unnecessary decimal places in decimal computations. For instance, in general it is unnecessary to carry computations with United States money beyond two places. Such problems also arise in measurements from the fact that exact measurements can not be made. We can measure a length only to within some fractional part of the unit employed, as the foot or the inch. Some measurements also are made much finer than others. For instance, machinists doing extremely fine work take account of $\frac{1}{1000}$ of an inch while the width of a street or lot is measured only to within an inch. The decimal places beyond those of which account was taken in the measurement are worthless. It is accordingly a waste of time to carry computations out for decimal places which will finally be thrown away. In order to find the amount of metal in a brass cylinder 4.00 inches long and .35 inches in radius the area of the cross-section must first be found. As the radius was measured to two decimal places only it will be useless to carry the volume to more than two decimal places. We first square .35 which gives .1225, and this result multiplied by 3.1416 gives a result of eight decimal places, of which only the

3.1416
.12
.31
6
.37

first two will have any value. The work is, therefore, contracted so as to give only as many decimal places as will have value in the final result. All such multiplications are begun with the first figure to the left of the multiplier. With a little practice this will be as natural as beginning with the first figure to the right. The process is as follows: multiply the multiplicand by the first figure to the left in the multiplier and write down as much of this partial product as will appear in the final result.

In this case it is $.1 \times 3.1416$ giving .31416, only the first two figures of which are written down, since but two decimal places are to be preserved. Had the first figure to be discarded been 5, or above, the next figure would have been increased 1 so that .32 would have been written in the place of .31. It is the practice of some in using the process of approximation to cut off from the multiplicand the figures which have not played a part in the partial product that is to be preserved. Thus in the present case, 416 would be cut off. A little practice will soon tell the operator which is the farthest figure to the right that will affect the final result, so that in practice, the total multiplication is never carried out. When the first partial product has been found, multiplication by each figure in the multiplier will start with the first figure in the multiplicand to the right which was not removed in the process of finding the preceding partial product. Unless multiplication of this figure farthest to the right gives a unit to carry, or a figure above 4, it will not affect the partial product. This is the case in the illustration given above; since $.1 \times .02$ produces no number to carry, nor is it above 4, hence the second multiplication gives only $3 \times .02$ or .06 to go down into the final result. Multiplication is continued in this way until the figures fall without the usable decimal places which happen to come after the second multiplication in the

illustration. Another illustration must suffice. Find the product of 32.14×5.26 to two decimal places.

32.14	32.14 × 5 = 160.70	— the first partial product.
5.26	32.1 × .2 = 6.43	— the product of .2 × .14 would have given three places and as the third figure was 8 it changed the next from 2 to 3.
160.70		
6.43		
1.93		
169.06	32. × .06 = 1.93	in the same manner as the last product.

Division can be approximated even more simply than multiplication by merely cutting off figures to the right of

```
3562)48276(13.554
     3562
     -----
     12656
     10686
     -----
      1970  (2 cut off from divisor).
      1780
      ----
       190  (6 cut off from divisor).
       175
       ---
        15  (5 cut off from divisor).
        12
        --
```

the divisor in the place of annexing ciphers to the dividend. The accompanying illustration is self-explanatory. If the divisor had contained only three digits, the division could have been carried to but two decimal places by cutting off figures only and we would have had to resort to the annexing of ciphers.

XV. EXERCISES

1. Show how decimals are an extension of the Hindu number-system.

2. Show the advantage of a decimal system similar to that suggested by Simon Stevin in which the unit's figure holds a separate position.

3. Give a brief résumé of the development and present status of decimals.

4. What is the effect upon the value of a number of moving the decimal point one figure to the right? two to the left? What is the effect of annexing a zero to .8? to 8.0?

5. Discuss fully the informal introduction of decimals during the first years, through United States money.

6. How would you introduce decimals formally as an addition to the Hindu number-system?

7. How would you introduce decimals formally as a new form of fractions?

8. Compare the two methods of introduction.

9. What common fractions can be reduced to decimals which do not repeat; i. e., which have for their denominators some power of 10? Illustrate.

10. Restate and solve in decimal notation the problems below under common fractions: 32, 33, 34, 35, and 36.

11. How would you go about making the idea of place value clear to a class?

12. Discuss the reading of numbers to be taught throughout the various grades.

13. How would you teach a class the location of the decimal point in addition? Illustrate.

14. Find the sum of $67.8\frac{1}{2}$, $7\frac{1}{2}$, $9.6\frac{1}{2}$, $15\frac{1}{2}$, $7.0\frac{1}{2}$, and $4\frac{1}{16}$ correct to .001.

15. Subtract by the Austrian method:

83.432	67.081
$12\frac{1}{2}$	$3.0\frac{3}{10}$

Give results correct to .001.

16. Compare finding the sum of

$23\frac{1}{2}$		23.33
$15\frac{3}{7}$	with finding the sum of	15.29
$19\frac{1}{3}$		19.13
$3\frac{2}{5}$		3.40

17. Outline by means of examples the order of taking up the difficulties in the multiplication of decimals.

18. What is the law governing the location of the decimal point in multiplication? How should a class be led to discover this fact?

19. How much more will it cost to send a box weighing 350 lb. by express at 60 cents per hundredweight than by freight at 30 cents per hundredweight?

20. If .25 be added to a certain number 15 may be subtracted 75 times. What is the number?

21. Outline by means of examples the order of taking up the difficulties in the division of decimals.

22. What is the law governing the location of the decimal point in division? How should a class be led to discover this fact? Show that this law could have been derived from that of multiplication by the application of the inverse process.

23. Give a short method of multiplying or dividing by 10 or any power of 10; by .33⅓, by .66⅔, by .125, by .06¼, etc.

24. Of what number is 25.56 both divisor and quotient?

25. The sum of the divisor and quotient is 414.06. If the divisor is .6 what is the dividend?

26. Criticize: $\frac{1}{6} \times \frac{1}{\frac{3\emptyset}{4}} \times \frac{.6}{3.6\emptyset} = \frac{.3}{4} = .75.$

27. If .75 of a number is 11.25, what is the number?

28. 544.932 is what decimal part of 612.3?

29. $6165 \times .83\frac{1}{3} = ?$

30. Express in a decimal the ratio of 243 to 540.

31. Explain how you would take up with a class the reduction of decimals to common fractions.

32. What two cases arise in the reduction of common fractions to decimals? When would you take up each? Explain your reasons fully.

33. Explain how you would take up with a class the reduction of each of these two forms of common fractions to decimals.

34. Change the following to decimals correct to .001: $\frac{1}{7}$, $\frac{3}{16}$, $6\frac{3}{11}$, $4\frac{5}{9}$.

35. What are the corresponding decimal forms of the following: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, $\frac{1}{25}$, $\frac{1}{50}$?

36. What decimal part of a ton is 8278 lb.? 6234 lb.? 12345 lb.?

37. Change the following to common fractions: .325, .03125, 8.52, .124.

38. What are "repeating" or "recurring" decimals? What is meant by $2.4315\dot{6}$? $.06\dot{1}7325\dot{}$? $.\dot{6}$? $.\dot{3}$? $.125\dot{}$?

39. Change the decimals in ex. 38 to common fractions.

40. Explain the finding of approximate results in multiplication by an example.

41. Do the same for division.

42. Approximate the following to .01:

i. $637.45 \div 6735$; iii. $8969 \div 243$; v. 3.1416×9.43;
ii. $47.5 \div 1.32$; iv. $.7638 \times 1.28$; vi. 46.78×3.472.

43. Find the area and circumference of a circle whose radius is 8.52 inches.

44. The freight rate in the United States averages \$.0047 per ton for each mile. Find to cents by approximation how much it would cost to ship 863.4 tons by freight 586 miles.

45. To how many decimal places should the area of a farm be found in acres in order that the price may be computed to cents, if the price is \$100 per acre? \$25 per acre? \$15 per acre?

46. The diagonal through the baseball diamond is $90 \times \sqrt{2}$ feet. To how many decimal places should $\sqrt{2}$ be computed in order to give the length of the diagonal in feet? in inches?

47. To how many decimal places must π be given in order to compute the circumference of a circle, whose radius is 8 feet, correctly to feet? to inches?

48. To how many decimal places should π be given in order to compute the circumference of the earth correctly to inches at the equator?

49. Suggest six problems in approximations suitable for grade work.

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CHAPTER XVI

METRIC SYSTEM

183. History. The metric system is the only system of weights and measures which has not sprung up by mere chance, but is the result of a scientific investigation directed towards a definite purpose. Taken with the Hindu numerals and decimal fractions it rounds out a unified number-system universally useful because of its simplicity, which makes an understanding of it comparatively easy.

Louis XVI of France in 1790 appointed BORDA, CONDORCET, LAGRANGE, LAPLACE, and MONGE to investigate the matter of selecting a more suitable system of weights and measures. They selected arbitrarily the $(10)^{-7}$ of the earth's quadrant along the meridian through Paris as the unit of length which they called *meter*. All of the other units were based upon this one thus arbitrarily selected, thereby establishing a system of reliable units. In December, 1799 this report was passed as a law in France, but was not made compulsory before 1837.

Scientific men throughout the world and all leading nations, excepting China, Japan, Russia, England, and the United States, have adopted the metric system. Since its adoption by the United States was first suggested by John Quincy Adams in 1821, several unsuccessful attempts have been made to secure its exclusive adoption. In 1866, the metric system of weights and measures was legalized but not made obligatory. The international electrical units, based upon the metric system, were made the legal electrical units in 1894.

The United States Army and Navy use the metric units exclusively, foreign postal rates are based upon metric weights, as is also the amount of silver in the half-dollar, quarter-dollar, and dime, even the yard is defined in terms of the meter, while in Porto Rico and the Philippines the total metric system is made obligatory.

Of late years, medical men and pharmacists have been active in urging the adoption of the metric system. Some manufacturers, as well, have awakened to the value of a system of weights and measures as simple and as universal as the metric system and are using it more and more as time goes on. From the above statistics it may be inferred that, in the United States at least, the adoption of the metric system to the exclusion of all other systems is only a matter of time.

184. Fundamental Problem. Since the adoption of the metric system is only a matter of time, it may well behoove the teacher to acquaint the pupils with the elements of this simple system of measures, so that a pupil who may be called upon to use it in the future will be able to recall the tables and their use in a few minutes. The main emphasis should, however, be placed upon educating the pupils to see the great advantages of this new system over the old English system. As an aid to the teacher in presenting this phase of the subject the following résumé of the advantages and disadvantages resulting from a change in the metric system has been prepared.

185. Disadvantages. The following disadvantages are advanced as arising from the adoption of the metric system:

i. *The present generation must learn a new set of tables and the use of these tables.* The extreme simplicity of the metric system, however, as will be explained later, makes this anything but a serious matter.

ii. *This change of units will produce confusion.* Com-

parison between similar units of the two systems will show how unfounded is this anticipated difficulty. Even though this confusion did result, it would be only temporary. In the first place, this will not apply to merchants and technical people who would learn the new system as they would learn a new price list, and so on. The average householder would find nothing confusing in the generally used units, since the meter is only 3 in. longer than the yard; the *liter* is nearly mid-way between the dry and liquid quart; one-half a *kilogram* is a trifle over a pound; and the *metric ton* is about 2200 pounds. Such a change would, therefore, not bother the average man or woman in the general business transaction involving these units. Moreover, there is the merchant who would act as a sort of guide during the first few months of the use of the new system.

One means of familiarizing the public with the new units and their relation to the old units would be to require the new system to be taught in the public schools for a stated time, say three years, before its adoption. Another means would be the distribution through the United States mail of a pamphlet giving the necessary information. Most of the difficulty would come not through a lack of ability to learn the new system, but through indifference and a desire to hold stubbornly to the old in spite of the many advantages of the new. The experience of other countries which have made the change has been most satisfactory and the old system has usually gone out of use in about a year.

iii. *All dealers would be obliged to reconstruct their prices.* While this objection is often raised it is really not worth considering as such adjustments are made several times a year in many establishments.

iv. *A uniform system should be in use by all of the English speaking peoples.* Such sentiment can only retard progress. Because we ride in automobiles we do not venerate

the less our forefathers who rode in ox carts. But even should we desire to respect this bit of sentiment, it would be impossible to do so since there is no common system of weights and measures among the English speaking peoples. For instance, the imperial gallon of Great Britain is to the standard gallon of the United States in the ratio of 277.274 to 231. It is fully as troublesome to change from the United States quart to the English quart as to the liter. Two measures are not equal just because they have the same name.

v. *The metric system is decimal.* The great superiority of a duodecimal over a decimal system is, of course, very apparent; but that is not the issue. The metric system was purposely made decimal in order to conform to the Hindu number-system. Can anything be further from a duodecimal system than such converting factors as $16\frac{1}{2}$, $5\frac{1}{2}$, 3, etc., of the English system?

vi. *The change will entail a great expense upon the merchants and upon the people in general.* In answering this objection, it must be remembered that the only real expense will be the comparatively small one of recasting the weights and beams of the scales. Spring scales need only be regraduated. Interchangeable use of the liter and quart could be permitted for two or three years as the two measures are so nearly equal, and during this time the manufacture of quart and gallon measures could be discontinued. The quart and gallon measures would thus eliminate themselves by the wearing-out process. The dry goods merchant would be required to change the brads in his counter so that they would show the meter and its fractional parts in place of the yard and its fractional parts. In addition it would be necessary to replace the stamp used to manufacture the advertisement yard stick with one for the meter. The real expense would be in changing the manufacturer's patterns,

which would be done without murmur if it were brought about as the result of a new invention. Again, many manufacturing plants of weights and measures carry patterns for both systems. To them the change would be an actual saving of expense.

vii. *The various manufacturers would suffer a heavy outlay occasioned by the changing of patterns, casts, machines, etc.* In the first place, all universal standards, such as railroad gauges, are not to be altered. Moreover, the most ardent, rational advocate of the metric system does not clamor for a too hasty or a too abrupt adoption of the new system. A long time limit could be placed upon plans, patterns, and so on, now in use, at the same time that all new designs, patterns, and so on, could be required to be made in the metric measurements. Even a limited amount of new designs might be permitted for the improvement of existing machines. In fact, the time limit on existing patterns could be wholly removed without invalidating the advantages of the metric system. Except in manufacturing plants, dimensions are seldom referred to, and our rapid changes would soon eliminate the old system even there. Had the metric system been adopted in 1890 with the view of making it wholly operative in twenty-five years, we would be using these measurements exclusively at the present time. Many industries which have since sprung up, such as the manufacture of automobiles and motorcycles, would have avoided the change. Hence, it would be very much better for this change to be made soon, even though it may not become completely operative for a long time. It is true, some loss is bound to occur through such a change. This loss will come, however, only once. If $12,000,000 could be expended upon the dreadnaught Moltke, which was able to produce destruction only, why may not a few dollars be expended upon a constructive change? While such losses are

generally borne by the individual, the expense in this case could be distributed by a slight national revenue tax so that it would be felt by no one in particular. The adoption of the metric system is merely a matter of time, as has been said before, and the sooner it is accepted the cheaper it will be for the manufacturers, since they will thus have fewer changes to make than they will after their plants have grown materially. Wise laws administered from a common-sense standpoint, in which much elbow room is permitted, are necessary and desired by all true advocates of the metric system.

viii. *The change in land measurements would cause inconvenience.* It is very much to be regretted that a decimal system of measures had not been adopted in this country at the time that the survey of the Northwest Territory was begun. The difficulties which will arise in connection with land measurements in the west by the adoption of the metric system are vital and the only ones of any importance. Even these difficulties are far from insurmountable, however, as it would result in only one inconvenient relation in the place of the many of the English system. If we carry the reduction of acres to hektares out to six decimal places we get one acre equal to .404858 hektares, thus making 160 acres equal to 64.78 hektares. This lacks only .22 hektares of being 65 hektares, a variation of a little over .3 of one per cent. Most of our surveys are made no closer than this to the true contents of a quarter section. While 65 is not as serviceable a number as 160 it is still usable. The land in the southwest was originally laid out on a metric scale. Its conversion to acre measure is not noticeable there now. Again, all irregular pieces, as are most divisions in the east, together with small measurements, as city lots, will be affected only slightly by this complete change. Records need be changed only as land is sold.

186. Advantages. The exclusive use of the metric system by men of science is a sufficient recommendation for its universal adoption. Our great scientists are all men of simple expression and of simple modes of thought. Simplicity is the fundamental principle of the metric system. Through its beautiful simplicity it embodies the essentials of a good system of weights and measures; it is easy of comprehension, compact as to extension, serviceable in application, and definite in meaning. As a nation, we are not keeping pace with modern thought in service and efficiency so long as we neglect the adoption of the metric system.

i. *Simple Tables.* The simplicity of the metric tables can best be shown by comparing any one, for example, the table for length, with the corresponding English table.

3 barley corns = 1 in.	10mm = 1cm
12 in. = 1 ft.	10cm = 1dm
3 ft. = 1 yd.	10dm = 1m
5½ yd. = 1 rd.	10m = 1Dm
320 rd. = 1 mi.	10Dm = 1Hm
	10Hm = 1Km

Any one who is not stubbornly clinging to the old customs can learn the metric tables in a few minutes. Excellent use can be made of the analogy existing between the United States monetary table, with which all are familiar, and the first part of the metric tables.

10 mills = 1 cent	10mm = 1cm
10 cents = 1 dime	10cm = 1dm
10 dimes = 1 dollar	10dm = 1m

Students wholly unfamiliar with the metric tables have been able to give any of them with little or no hesitation by the aid of this analogy after having heard any one of the tables. The children of our schools could learn these new tables in about one-tenth of the time that they now require

to learn the English tables. They would also know them more accurately and have a clearer comprehension of their meaning than they have of the English tables. And surely there is no one, knowing the English tables, who would refuse to sacrifice a few minutes in learning new ones in order that posterity might have this simple, time-saving device.

ii. *Compactness of Tables.* The tables for square and cubic measure differ only slightly from that for length, while those for weight and capacity are exact duplicates. These tables contain only one constant, 10, and two of its powers. If *stere,* cubic meter used in measuring wood, *hektare,* square hektometer used in measuring land, and the *metric ton,* 1000 Kg. are included, the list of terms will be complete and will comprise only fourteen in all. The range of each table is such as to make several tables unnecessary. For instance, delicate amounts, like certain drugs, are weighed in milligrams; more ordinary drugs and jewelry in grams; groceries and general weights in kilograms; and larger amounts, such as coal and hay, in the metric ton. Thus one table is used for weights ranging from a hair to a ton of coal.

The value of this compactness can hardly be overestimated. In comparison with the English system of many tables, a host of unrelated terms—there are over fifteen for linear measure alone—and an innumerable collection of heterogeneous constants, we have only five tables built upon a single principle which conforms to our decimal number-system, only fourteen terms, and just one constant together with two of its powers. It is, indeed, an admirable illustration of *multum in parvo,* and it is this quality that enhances its practical value.

iii. *Definiteness of the Units.* Four liters is perfectly definite and means the same thing the world over. Four

quarts may have at least three different meanings. Ton, ounce, and so on, also have different meanings. In order to simplify our increasing foreign trade, if for no other reason, the United States should adopt the metric system. As the metric system has no unit corresponding to the bushel, such articles as grain, which are usually sold by the bushel, must be sold by weight after the adoption of the metric system. The great discrepancies existing between the States, by reason of their varied definitions of bushel in terms of weight, will thus be automatically eliminated. Weight, not bulk, is the only true measure of quantity in such cases, as is being put into practice more and more in the United States, especially in the west. Why complicate matters by quoting the price upon 32 lb. of oats, 56 lb. of corn, and so on, instead of upon 100 lb. of each?

iv. *Simple Application of the Tables.* The simpleness of the metric tables makes them not only easy to learn but yet more easy to apply. This is clearly shown by comparing the reduction of 6 Km. 5 Hm. 7 Dm. 3 m. 8 dm. 9 cm. 5 mm. to meters, to centimeters, or to kilometers with the reduction of 3 mi. 154 rd. 4 yd. 2 ft. 7 in. to feet, to rods, or to inches. Gunter recognized the value of a system of measures based upon decimals in the construction of his measuring chain, which has been used by surveyors for many years because it makes "the work more easy in arithmetic." For this reason, it is being adopted in many professions and manufacturing establishments, at the present time.

The relations existing between the units of the various tables are also very important. Only one unit—length—was selected arbitrarily and all of the others made to depend upon this one. The meaning thus given to the various units is an inestimable aid in such scientific measurements and computations as specific gravity.

v. *Metric System and Decimal Fractions.* The weights

and measures of any country form a part of its number-system. Adoption of the metric system will, therefore, bring decimals into the foreground and will make common fractions the *un*common form of fractions. The advantages of decimals over common fractions need not be gone into here. It is merely desired to point out the fact that the elimination of common fractions with the adoption of the metric system will mean another great saving of time as well as a great simplification of our number-system.

187. Development of the Work. In introducing the metric system to a class, let the teacher base it upon the similarity of the metric tables to that of U. S. money. The first day, the linear table involving fractional parts, only, is to be learned and applied to several simple problems in reduction graduated in a fashion similar to that suggested under the English system. The linear table may be completed the second day with some more difficult examples. For the purpose of comparison, a few problems like the following are suggested: reduce 34 rd. 5 yd. 2 ft. 7 in. to inches; to feet; to rods.

Next should come tables of square and cubic measure to be worked out by the class without reference to the text. After the teacher has explained the relation of the liter and gram to cubic measure, the tables of capacity and of weight should be given by the class, again without reference to the texts. Several problems giving practice in these reductions and showing the relations between the various tables will naturally follow.

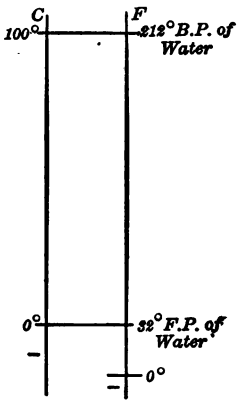
To make this matter clear as well as interesting to the class it is necessary that the school possess a liter measure, a 100 or 500 gram weight or a set of weights, and a meter stick, preferably with the metric system of measurements on one side and the English on the other. It may be interesting to note that our 5c piece is nearly 2 mm. thick,

2 cm. in diameter, and weighs about 5 g. Only three simple equivalences are necessary for making the changes from

1 m.	= 39.37	in.	nearly.
1 l.	= 1	qt.	"
1 kg.	= 2.2	lb.	"

English to the metric system, or *vice versa*. Therefore only a few problems involving these conversion factors need be given.

188. Thermometry. The study of measures is incomplete unless it includes a consideration of thermometers which



measure the degree of heat of a body. As the centigrade scale is coming into use more and more, this consideration may well include a comparison of it and the fahrenheit scale. This comparison is shown very clearly by the accompanying figure which is suggested for use in all conversions from one scale to the other. The attention of the pupils

should also be called to the simplicity of the centigrade scale over that of the fahrenheit scale.

XVI. EXERCISES

1. Give a brief historical review of the metric system.

2. Compare the development of the metric system of weights and measures with that of any other system.

3. Define meter, liter, gram, yard, quart, pound.

4. What is the present status of the metric system in the United States?

5. Show that the metric system, taken with the Hindu numerals and decimal fractions, form a complete number-system.

6. What should be the teacher's chief aim in teaching the metric system? How can this be accomplished?

7. State and discuss the disadvantages mentioned which would result from the adoption of the metric system, and any additional ones which may occur to you.

8. How long does it take you to reduce

i. 3 mi. 125 rd. 4 yd. 2 ft. 7 in. to inches? to yards? to miles?

ii. 4 Km 5 Hm 7 Dm 8 m 2 dm 5 cm 3 mm to millimeters? to meters? to kilometers?

9. Explain the use of the United States money table in teaching the metric tables.

10. Compare the following computations in finding prices:

i. The price of 4 yd. 2 ft. at 45 cents per yard with that of 5 m 7 dm at 52 cents per meter.

ii. The price of 5460 lb. at \$7.50 per ton with that of 2640 kg at \$7.90 per metric ton.

iii. The price of 5 gal. 3 qt. at 7 cents per quart with that of 6Dl 5 l at 7 cents per liter.

11. What various meanings may be attached to the following: ton, ounce, mile, pace, fathom, quart, bushel of oats, gram, liter, meter?

12. Find the specific gravity of $1\frac{1}{2}$ cu. ft. of cast iron weighing 532.5 lb. Find the specific gravity of 1000 cc of cast iron weighing 7.3 kg.

13. What is the weight of a quart and of a liter, each, of mercury, copper, cork, and brass?

14. The specific gravity of ice is .92. Find the weight of 1 cu. ft.; of 1 cu. yd.; of 1 cu dm; of 1 cu m.

15. What are the approximate metric equivalents of 15 lb.; 4 gal. 3 qt.; 6 yd.; 4½ ton?

16. State and discuss the advantages which would result from the adoption of the metric system. Consider additional ones than those mentioned.

17. Change, by use of a figure, 68° F to C; 5° C to F; 16° C to F; —4° F to C; —5° C to F; 12° F to C; 90° C to F.

18. What is the superiority of the centigrade over the fahrenheit scale?

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CHAPTER XVII

PERCENTAGE

189. History. Percentage is the business man's arithmetical language. He does not say that the output of his factory has increased $\frac{1}{4}$ of that of last year, but that it has increased 25%. This custom of counting by hundredths is an old one. The Italian arithmetics of the 15th century speak of "per cento" for which the symbols "per C" and "C" are used. By the 17th century they wrote "p$\frac{c}{o}$" from which our universal % has likely arisen. Whether the terminology of percentage is an improvement over that of common fractions and decimals, whether it is easier or simpler to say thirteen per cent or thirteen hundredths, need not be entered into here. Since percentage has universal business usage it has rightly been given a place in the course of study.

190. Introduction. The teacher can eliminate many of the dreadful hobgoblins which lurk in the new words of the subject by a simple incidental introduction. Some little time, say a week, before the formal study is undertaken let him mention in connection with problems containing such decimals as .24, .08, .45, etc., that the business man refers to these as 24 per cent, 8 per cent, 45 per cent, etc. In case the text does not contain an ample supply of such examples, the teacher can formulate them very easily, either by changing the numbers used in the text or by making entirely new ones. Most of the pupils have heard men speaking of per cent or of percentage, hence these terms will

not be wholly new and should create no bewilderment when mentioned thus incidentally. As these terms form part of the language of men engaged in actual business the pupils will be interested in becoming familiar with them and their use. Interest may also be aroused in percentage by requesting the pupils to report from day to day any expressions they may have read or heard containing these terms.

Furthermore, they can be encouraged to make use of the expression per cent when speaking of decimals; and, in a few days, may be asked to re-state in the percentage terminology, some problems they are solving under the decimal notation. It may also be well to reduce some simple common fractions and decimals to common fractions with the denominator 100, at this time. This should be made to evolve into the more formal work of the same nature to be considered later. In order to make this informal introduction effective the teacher must recognize its purpose clearly and not thwart the same by introducing such technical terms as percentage, base, rate, and so on, however tempting this may be. The term *base* is not used in business at all.

191. The Teacher's Object. The difficulties which arise in the study of percentage are almost wholly difficulties of language,—difficulties of expression. All of the arithmetical principles involved have arisen before in the study of common fractions and decimals. The forms $\frac{1}{5}$, .2 and 20 per cent are equivalent expressions. It is only the language, the terminology, which is new to the pupil taking up the study of percentage. His difficulties are those of thinking in this new language, which is as truly a new language to him as is Latin to the beginner of that subject. Recognition of these facts makes it much simpler to formulate the teacher's mode of attack, to foretell the pupil's difficulties, and to suggest remedies.

Most teachers recognize, in a sort of way, that the difficul-

ties of percentage are those of language; but as a rule, they are either unable to find a method of approach from this standpoint, or make no attempt to do so. Though percentage presents no new arithmetical principles, it is one of the poorest taught subjects in the whole school course, the chief reason being the misdirection of emphasis by the teacher.

192. Teaching the Language of Percentage. Learning the language, then, is the first requirement in the study of percentage. This is not to be an etymological discussion of the terms used, but a continuation of the informal introduction outlined above which will lead the pupils by easy stages to an understanding of the expressions and their correct use. One phase of this work will be the establishment of the connection between the fractional, decimal, and percentage forms of expressing a number. A good way will be first to reduce common fractions to decimals and then to re-state them in the percentage language. Decimals may next be reduced to common fractions and to percentage, after which a number expressed in the percentage notation may be reduced in turn to decimals and to common fractions. Daily drill work in supplying the missing parts in such columns as the following, will be found very helpful:

$\frac{3}{5}$ =	. ? =	?%
? =	.24 =	?%
? =	. ? =	52%

The numbers used at first should be such as can be reduced to common fractions having the denominator 100. When these reductions can be made readily, an advance step is taken by introducing numbers which will result in frac-

tional per cents, as, $\frac{1}{6}$, $\frac{1}{8}$, etc., and finally numbers which are over 100%, as, $1\frac{2}{3}$, etc. This work must not come all at once at the beginning, however, and should be distributed throughout the early periods of the study. The numbers used in these drills must be the same as those entering into the problems.

193. Application Phase of the Language. A second phase of the language work in percentage is the application of these number relations to problems. This may be introduced, as previously mentioned, in the informal lessons and by the assignment of problems solved previously under fractions and decimals for translation into the language of percentage. At first these examples need not be solved as percentage problems, but merely be stated as such.

The process is now reversed; problems stated in the percentage notation are re-stated and solved as problems in common fractions and decimals. A sufficient quantity of this work should be done to enable the pupils to recognize off-hand that sets of three problems, such as the following, are not different problems but only different statements of the same problem:

1. Goods valued at $1200, after being damaged by fire, were sold at a reduction of $\frac{1}{4}$. For what were the goods sold?
2. Goods valued at $1200, after being damaged by fire, were sold at a reduction of .25. For what were the goods sold?
3. Goods valued at $1200, after being damaged by fire, were sold at a reduction of 25 per cent. For what were the goods sold?

Even after this work, the new language will be far from mastered. This will come only through eternal vigilance on the part of the teacher. The object of problems selected for explanation, and of all examples for drill at this time, should always be to help the child to the acquisition of this new language and facility in its use. To succeed in the teaching of percentage, it is necessary that the teacher never

lose sight of the fact that the difficulties are mostly those of language. He must ever be on the alert to see that the clearness of the work is not clogged because the language is not understood.

194. The Fundamental Problem. Most of the problems in percentage deal with finding a given per cent of some number; hence, this point should receive careful attention. Their simplicity makes them excellent material through which to learn this new arithmetical language. There is grave danger, however, that the teacher may fall into the error of relaxing the attention just because the problems are simple. Their importance makes it necessary not only that a great number be solved, but also that they be thoroughly understood.

Variety in the statement of the problem is absolutely necessary, so that such forms as (5), (6), or (7) given below, and others, be used as freely as form (4) which is often made the standard to the exclusion of all others. We are here encountering a more general form of language lesson than the special one connected with the subject of percentage, but nevertheless a most important one.

4. What is 20% of the salary of a man who receives $1500 per year?

5. How much does a man save who spends 80% of his salary, when it is $1500 per year?

6. Two automobiles start from the same place and go in the same direction, one at the rate of 25 miles per hour and the other 20% faster. How far will they be apart at the end of 6 hours?

7. How many problems does a pupil solve who fails to solve 20% of a total assignment of 15 problems?

8. What will a man's yearly income be after his present income of $1500 has been increased 20%?

195. Variety of Form of Statement Important. It seems natural for the pupils to solve problems similar to the last one above by finding the amount of increase or decrease

and then adding to or subtracting from the original amount. While such solutions are perfectly valid it will expand the pupils' knowledge of the subject if they are taught to recognize that the amount sought is 100 per cent plus the rate of increase, or minus the rate of decrease of the original amount. Fractions or decimals can be used to make this matter clear. In the above we would find that the man's income after the increase would be $1\frac{1}{5}$ or 1.2 of what it was before.

196. Per Cents of Per Cents. Problems on finding what per cent a certain per cent is of another, as, "What is 20% of 15%?" are liable to prove confusing at first. Most of this confusion may be eliminated by the introduction of the matter through a few well selected problems in common fractions, as in the following:

9. A man owning $\frac{1}{2}$ of a section of land sells $\frac{1}{4}$ of it. What part of the section does he sell?

The translation of such problems, and their solution in the language of percentage, is continued until the pupils recognize that a per cent of another per cent means a per cent of the whole just as is the case with common fractions.

197. Special Per Cent-Forms. In order to keep the language of percentage clearly in sight all of the time, a few problems should be stated and solved by the use of common fractions and decimals exclusively. All problems containing a per cent which is an aliquot part of 100%, as $12\frac{1}{2}$%, $16\frac{2}{3}$%, etc., should always be solved as problems in common fractions. It is also important to notice that in all problem work, the per cent is used in decimal form.

198. The Inverse Problem. As in common fractions and decimals the inverse problems generally give the most trouble, so they do in percentage. For example:

10. A man saves $325 yearly, which is 15% of his salary. What is his salary?

The solution of this and similar problems, however, will present no special difficulty to pupils who have been taught the idea of the inverse processes and have been given sufficient practice in their application. To them the $325 is the product of two numbers, one of which is .15 and the other unknown which they find by dividing the product by the given number, .15. The literal equation is an excellent instrument for expressing the relation between the numbers in percentage, as

$$N = n\,(?\%)$$

and for showing what operation is necessary to find any number in terms of the two given ones. For the pupils unacquainted with the literal equation the scheme for showing the relation between the numbers, as previously suggested, will be found useful with certain pupils, and especially at first. For the problem in question this takes the accompanying form, in which the dots are to be supplied by the proper number to make a correct multiplication. The use of the reciprocal idea will be found the natural method of solution. We have here one simple relation connecting all of the numbers involved, in the place of three isolated *cases* of the old texts for finding percentage, base, and rate. A clear understanding of the subject can not be had unless this listing of problems under cases be discontinued. The simple unification of the process expressed by one relation has everything to commend it over the several isolated cases; it is understood much more clearly and quickly, and can also be retained in the memory much more easily.

———
.15
$325.00

199. Problems for Socializing Percentage. Baseball and football were activities mentioned as useful aids in

socializing arithmetic. Ranking of teams, individual fielding and batting averages are expressed as three figured decimals but are always referred to as per cents. For instance, a baseball fielder who has accepted 24 out of 26 possible chances would have an average of $\frac{24}{26}$ or .923. This is spoken of erroneously as "923 per cent," but is, of course, in reality 92.3 per cent. Because these averages are universally spoken of as per cents and further because such computations involve both the reasoning and sports interest of somewhat older children this work is suggested for the

Name	A. B.	H	A	P	E
H. Drake, ss...	4	2	5	3	1
J. Adams, c....	5	2	3	8	0
F. Jones, 3b...	3	1	4	2	0

topic of percentage rather than for that of decimals. The three problems mentioned above furnish most excellent material for finding rates. Every boy will be interested in computing his team's ranking, as well as his individual batting and fielding average. A successful use of this material requires that the teacher be informed regarding our national game. Not only a general knowledge and interest in the game itself is necessary for this purpose but ability to read a "box score" and to use the same in the computation of the various averages.

200. Percentage as Unitary Analysis. Unitary analysis, or the so-called hundred per cent method, is often used

in the solution of the inverse problems. By this treatment, the preceding problem would become,

15% of a man's salary = \$325,
1% of a man's salary = $\frac{1}{15}$ of \$325, or \$21$\frac{2}{3}$,
Hence 100% of a man's salary = \$21$\frac{2}{3}$ × 100, or \$2166$\frac{2}{3}$.

This form of unitary analysis may be made quite serviceable in introducing the subject and later on in making complete explanations, when these become necessary. After the process is understood, however, it should be abbreviated to one multiplication by $\frac{100}{15}$, in the place of a separate division and multiplication, or to division by .15, as was done in applying the idea of *reciprocity*.

201. Avoid a Common Corruption of Unitary Analysis. Care must be taken never to use, or permit the pupils to use, the following corruption of the solution by unitary analysis. In addition to containing a superfluous first statement it is incorrect and misleading to the pupils.

Let 100% = the total salary,
∴ 15% = \$325,
∴ 1% = \$21$\frac{2}{3}$,
∴100% = \$2166$\frac{2}{3}$.

The sign % is used as a symbol for the man's salary, a meaning which it was never intended to give it. Per cent, or its symbol %, is always used in the sense of a fractional part of some number or quantity, and never as a separate, intrinsic thing existing by itself. Hence % can not be used as above. The question involved is not one of mere notation, but one of fundamental ideas; one of true thinking.

202. Avoid Excessive Analyses. As in the case of the direct problem, many different forms or varieties of statement must be used. This brings us back to the original proposition of looking at the work in percentage largely as a language study. Moreover this manner of treating the

subject must not end here, but must be applied throughout all later lessons. On the other hand, the teacher should be careful not to require the pupils to learn and explain minutely questions of language which they understand thoroughly. Solutions of inverse problems should be made mental operations as soon as the language and forms of solution are understood, just as in any other form of number work. The literal equation, and other similar devices, should be discarded as soon as the pupils can do their work without them.

XVII. EXERCISES

1. Explain and illustrate that decimals and percentage are each a fractional notation just as truly as are common fractions. What class of people in the United States use each?

2. Explain fully and criticize the incidental introduction mentioned under 190.

3. Suggest some other form of introduction. What are its strong and what its weak points?

4. What restrictions must be placed upon denominators of fractions in order that they may be reduced to others with denominators of 100 and integral numerators? Suggest at least ten such fractions.

5. Change each of the following to the other two forms of fractions, placing the results in a table as in 192: $\frac{1}{2}$, .145, $12\frac{1}{2}\%$, .1345, 145%, $\frac{7}{8}$, $33\frac{1}{3}\%$, $.56\frac{1}{2}$, $3\frac{1}{4}$, .126, 245%, 16.5%, 2.45, $15\frac{3}{4}\%$, $5\frac{3}{8}$, 24.8%, $.06\frac{1}{4}$.

6. Suggest a list of three fractions of each kind for the formation of a table similar to that in 192 by pupils beginning the study of percentage. Also suggest a similar list of more difficult forms for further drills. Give the other two forms for each. When and how should these be used in the classroom?

7. What problems stated in the percentage notation should always be solved as problems in common fractions? Illustrate.

8. Select from an arithmetic a list of problems in common fractions and decimals suitable for translation into the language of percentage. Make the translation for at least six problems and solve, using the percentage notation.

9. Select from some text six problems of the type of 4 and change them to forms 5 and 7 found in 194.

10. Show that in all problems on percentage the decimal form of the fraction is always used. Illustrate.

11. What recognition must teachers take of this in teaching the subject?

12. Solve the problems under 194, 196, 198.

13. A man bought an automobile for $750 and sold it for 40% of that price. What was the average yearly depreciation if sold after being used six years? five years?

14. If 95% of the business men of this country are failures, how many are successful in a town of 80 business men? of 56? of 124? of 364?

15. The output from a factory is $30,000 the first year, 15% more the second, and 15% less the third year than the second. What was the output the third year? What difference would it have made if the decrease had come the second and the increase the third year?

16. A man owing $1400 decreases his indebtedness 20% each year for three years. How much does he owe at the end of that time?

17. 540 is what per cent of 3600? of 360? of 36? of 180? of 1800? of 600? of 60?

18. What is meant by some per cent of another per cent? Illustrate.

19. What is $\frac{1}{2}$ of 45%? 6% of 15%? $16\frac{2}{3}$% of 24%? 15% of .81%? 25% of 60%? $\frac{2}{3}$ of 75%? .2 of 65%?

20. The output for the third year of the factory mentioned in problem 15 was what per cent of that of the first year in the two cases? Solve by using the $30,000; by not using it.

21. The man mentioned in problem 16 owes what per cent of his original indebtedness at the end of the three years?

22. John solves 90% of an assignment of problems and Henry solves only $\frac{2}{3}$ as many. What per cent of the total assignment does Henry solve?

23. The increase in population in a certain town from 1900 to 1910 was 32%. The population in 1910 was what per cent of that in 1900? The population in 1900 was what per cent of that in 1910?

24. Solve and explain as to a grade class, problem 9 in 196 by means of a diagram for the purpose of conveying to the pupils the idea of a per cent of a per cent.

25. Suggest a list of problems similar to Ex. 22 for grade work. Solve and explain each.

26. A cow gives milk which is 3.8% butter fat. One pound of butter fat produces 1¼ pounds of butter. How many pounds of butter will 100 pounds of milk produce? What per cent of the milk will make butter?

27. A factory through improvements on its engines increased its efficiency 15% and by installing smoke consumers, decreased the fuel bill 6%. What per cent of the original fuel bill was saved?

28. Find the "percentages" of the following ball teams: A, won 12, lost 8; B, won 8, lost 14; C, won 96, lost 73.

29. Compute the batting and fielding averages of the players who appear in the "box score" under 199.

30. What class of problems in percentage can be solved more simply in terms of common fractions? Illustrate.

31. Change the numbers in problems 13 through 17 so as to make them problems for common fractions.

32. Suggest at least three other such problems for grade classes.

33. Explain what is meant by direct and inverse problems. Illustrate.

34. Mr. A's salary this year is $1800, an increase of 20% over what it was last year. What did he receive last year?

35. What is the percentage of attendance for a month of four weeks in a school with an enrollment of 36 pupils and which has had 42 half day absentees?

36. In a town election, candidate A received 465 votes and his competitor, B, received 390. What per cent of those voting for A could have voted for B, and what per cent could have remained at home, and A would still have been elected?

37. What is meant by "90% medicine"? How much water should be added to make it 45%? 80%? 40%?

38. A teacher marks each question in a test on a scale of 10. How can the sum total be changed to percentage if the number of questions is not 10? Illustrate for the cases of 8, 12, and 15 questions.

39. A man has 75% of his farm in cultivation and the remainder in pasture. What per cent is in pasture? The pasture is what per cent of the cultivated land? The cultivated land is what per cent of the pasture?

40. Solve problem 39 after replacing 75% by 60%; by 80%.

41. Suggest a list of six problems in percentage which you have previously found difficult or which you expect your future pupils to find difficult. Let these be tabulated, solved, and discussed by the class.

42. Why is it not a good plan to divide percentage problems into the three cases of finding percentage, base, and rate?

43. Explain the use of the literal equation in solving problems in percentage.

44. Show how the literal equation embodies the idea of the inverse process. Illustrate.

45. Explain the use of the forms suggested for classes unacquainted with the literal equation. Illustrate.

46. Solve problem 34 by unitary analysis. Show the advantages and disadvantages of this method.

47. Discuss the criticism of the form given under 201.

48. Illustrate by means of problems that the difficulties of the average pupil in percentage are difficulties of language.

49. Select data from the World's Almanac or other sources, suitable for problems for grade classes. Tell in what part of the work each is to be used.

CHAPTER XVIII

APPLICATION OF PERCENTAGE

203. Motivation through Activities. One of the most important parts of the whole subject of arithmetic is the business application of percentage; not so much as an application of percentage but rather as a study of business procedure. The effectiveness and practical value of this work, however, will depend in large measure, first, upon how real it is made by the teacher; second, upon how thorough a study is made, as each topic is taken up, of the terms and business transactions which enter into it; the reference to actual business practices, the samples of the various business papers and documents used, the writing and filling out of such papers by the pupils, visits to the various business enterprises in the vicinity, and the introduction of problems and business transactions dealing with the special industry of the community. It is well to encourage the pupils to engage in mock business activities from that of playing store—the delight of all children—to operating a school bank.

In order to make this work real to the children it must conform to actual business conditions. For instance, if it is desired to give the children practice in finding interest at 8%, the note must be written in a state permitting this rate or it should be understood as 6% interest and 2% commission for making the loan.

Here is also the place for bringing up the various short cuts common to commercial computation. They will appeal

to the pupils and may be made the basis of number drills at this time. The division of topics found in this, and in the following two lessons, is made strictly for convenience. It is not, however, a suggestion for use in a grade course.

204. Commission and Brokerage. Problems arising under this topic are quite simple. Apt illustrations for successfully introducing the subject are plentiful. An explanation on the part of the teacher of the workings of the stock exchange, board of trade, and commission markets will be useful and interesting. The children may also be asked to get more information on these subjects both from the encyclopedias and from visits to these places, if they live in a city. The use of data and procedures not found in business practice is the chief thing to guard against. The following are the most common commission charges:

1. Buying or selling bonds and stocks: $\frac{1}{8}\%$ of par value.
2. Buying or selling grain by the carload: $\frac{1}{8}$c per bushel.
3. Selling cattle: 60c per head up to a maximum of $15 per carload.
4. Selling hogs and sheep: $10 per carload.
5. Selling hay by the carload: 50c to $1 per ton.
6. Selling fruit and other produce: no uniform rate.
7. Selling real estate: Fixed by law in many states at about 2% to 3% and increased by agreement. In some cities charges for selling cheaper vacant lots are above that of the more expensive property. Land is often sold at a commission of $1 per acre. Sometimes the owner agrees to take a certain "net" sum for his property. The purchaser rarely pays commission.
8. Securing teacher's positions: 5% of first year's salary.
9. Selling merchandise in stores: always low where given, 1% to 3%.
10. Selling merchandise as agent: generally high, sometimes reaching $33\frac{1}{3}\%$ to 40%.

205. Discount. Goods are often sold below their original market price; there are a number of reasons for this. Goods may be out of season or damaged, the purchase may be exceptionally large, payment may be made within a

certain short time, or a wholesale firm issuing catalogs may wish to keep prices to the retail merchant private and also be able to change these prices easily and cheaply. Two classes of problems arise in discount. The first are those depending upon the following relation:

$$\textit{Selling price} = \textit{marked price} \times (1.00 - \textit{disc.}) \quad (1)$$

The second are those arising from a series of two or more discounts. Private marking systems and their use are of special interest in this connection.

206. Profit and Loss. Merchants calculate profits and losses in per cent based always upon the *buying price*, unless otherwise stated. The fundamental equations for a sale at a profit or loss are:

$$\text{Gain or loss} = \text{buying price} \times \text{gain (or loss)\%, and} \quad (1)$$
$$\text{Selling price} = \text{buying price} + \text{gain, or} - \text{loss, or} \quad (2)$$
$$\text{Selling price} = \text{buying price} \times (100\% + \text{gain\%})$$
$$= \text{buying price} \times (100\% - \text{loss\%}). \quad (3)$$

207. Insurance. Insurance is, today, taken advantage of by all men, rich and poor. Hence, it should be included as a part of the business side of arithmetic even though the problems which arise are very simple. Years ago, men banded together to help share each other's losses caused by sickness, fire, loss of live stock, and so on. From this spirit of neighborly kindness has sprung the mutual insurance companies prevalent in rural communities at present. It will be well for the teacher to explain the principle of a mutual company together with any of the following forms which may not be contained in the text: fire, marine, tornado, crop, live stock, burglary, plate glass, and accident. In cities and most towns each business building has its own special fire rate and can be insured for only a short term of years, usually less than three years. The rate depends upon several things as, construction, proximity to inflammable

buildings, use of building, its private fire protection, and so on.

Farm and dwelling properties are insured against fire for three and five years at $2\frac{1}{2}$ and 4 times the rate for a year. Stocks of merchandise are rarely insured for more than one year because of their varying values. About two-thirds or three-fourths of the actual valuation is generally recommended as the proper amount of insurance to carry and that fact should be taken into consideration in the solution of problems on insurance.

It may also be well to note that, as a rule, companies do not carry large fire risks on any one building alone, but reinsure such in other companies. The teacher must not lose the opportunity to point out in this connection our extreme carelessness as a nation in the matter of fires. In 1915 the fire losses in the United States amounted to $172,033,200, or about $1.71 per capita as against about 33 cents per capita for Europe. The necessity of fire protection should be noted carefully.

As the other forms of property insurance are not nearly so common as fire insurance and as the conditions under which they are taken out also vary greatly, these can not be gone into here.

208. Sick Benefits and Life Insurance. Sick benefits and the following three most common forms of life insurance may also be explained if the teacher thinks best:

1. "Straight Life," in which the insured makes regular, definite payments throughout life.

2. "Ten, fifteen, and so on, Payment Life," in which the insured makes regular payments for the number of years specified. At the end of that time he will have a "paid up policy" which keeps him insured without further payment or he may cancel his policy and receive about 50% of its face value.

3. "Endowment," in which the conditions are like those in (2), except that premiums are higher and if the policy is canceled after all the payments have been made, at which time the full face value will be paid the insured. The amount of premiums on a policy depends upon the age at which the insured took out his policy.

209. Revenues. The teacher may well introduce this subject of revenues by calling attention to the need of funds for carrying on public improvement and service. It is also well to distinguish between internal revenues, income tax, inheritance tax, and customs collected by the national government, and the direct tax levied on real estate and personal property by the more local authorities of state, county, and municipality. As a rule, revenue problems would better be confined to customs, and real estate and personal property taxes. The first problem is to find the tax rate, technically called "spreading the tax," and the second, that of finding the amount of tax to be paid upon a certain assessed valuation.

XVIII. EXERCISES

1. Explain the importance of the applications of percentage.
2. Give a list of all the possible agencies you can think of for making this work real to your pupils.
3. Give two illustrations each of problems (a) conforming to business conditions and (b) those not conforming to business conditions, (sometimes found in the texts).
4. Find the following products:

a. $412 \times 233\frac{1}{3}$.	b. $618 \times 516\frac{2}{3}$.	c. 562×98.
d. $916 \times 9\frac{1}{4}$.	e. $642 \times 10\frac{1}{4}$.	f. $624 \times 19\frac{1}{2}$.
g. $160 \times 22\frac{1}{2}$.	h. $160 \times 11\frac{1}{4}$.	i. $612 \times 16\frac{2}{3}$.
j. 125×816.	k. $632 \times 16\frac{2}{3}$.	l. $412 \times 47\frac{1}{2}$.

5. Suggest six similar examples in short cuts for grade work.
6. Outline an introductory lesson on commission and brokerage.

7. Describe the stock exchange, board of trade, and the work of various brokers and commission men.

8. What would you pay for 3000 bushels of wheat if the market price is 98¾ cents per bushel? What did the seller receive for it? What did the commission man receive?

9. A real estate man sold a house for $3550. What was his commission at 3%? What per cent of the sales price did the owner receive?

10. A real estate dealer agrees to sell a house to net the owner $4500. For what must he sell it to give himself a commission of 5% on the selling price? What will his commission be?

11. A produce commission merchant after deducting his commission of 8% remits $289.80 to the consignee. What was the total sale?

12. Suggest two problems on commission or brokerage, having a local application, for a grade class.

13. Outline an introductory lesson on discount.

14. Discuss the reasons for giving discounts in commercial transactions.

15. What must a suit of clothes be marked to sell for $30 when a discount of 25% has been given? 20%? 16⅔%?

16. How must dining-room chairs bought at $5 each, be marked in order to permit the dealer to give a discount of 25% and yet make 20%?

17. Suggest for solution by a grade class an illustration of each kind of problem depending upon the discount relation found in 205.

18. Discuss fully the use of the fundamental discount equation in 205 in the solution of problems.

19. Find the single discount equivalent to the following:

a. 30%, 20%, 10%.	b. 20%, 10%, 30%.
c. 10%, 20%, 5%.	d. 25%, 20%, 10%.

20. A man bought some goods at $60 and marked them 100% above cost. What did he get for them if he sold them at 50% below the marked price? 25% below marked price?

21. A house costing $2000 was sold at an advance of 25%, which was 16⅔% below the asking price. What was the asking price?

22. Explain and illustrate what is meant by computing the profit and loss upon the buying price.

23. What is the gain per cent in buying peas at $1.90 per bushel and selling them at 10c per quart?

24. A house is bought for $3000. What is the gain or loss per cent in selling it for $2800? for $3200?

25. Explain the fundamental equations and illustrate their use in the solutions of the various problems which arise in profit and loss.

26. Two houses were sold for $2400 each. 20% was gained on the one, 20% was lost on the other. What was the total gain or loss?

27. A scale weighs only 15 ounces to the pound. What is the merchant's illegal gain per cent? What is the customer's loss per cent?

28. A farmer is offered 80 cents per bushel for his corn in the fall but decides to hold it for a higher price in the spring. If the corn shrinks 13% what must he receive for it 6 months later in order not to sustain a loss? (Do not take account of interest.)

29. A dealer's stock invoiced at $4500 January 1. During the year he bought goods to the value of $9600 and sold goods for $13,460. His yearly expenses were $2300 and the invoice at the end of the year $4845. What was his per cent of profit on the sales? on the investment?

30. Mr. Jones bought a house for 10% below the original asking price and sold it later for 10% above the original asking price, thereby gaining $600. What did he pay for the house and what was his gain per cent?

31. At what price per acre will a real estate dealer's commission of 3% or $1 per acre be equal?

32. What is the gain per cent on goods sold at wholesale catalog prices which are bought by the retail dealer at a discount of 20%? of 35%?

33. Give illustrations suitable for a grade class of the various kinds of problems which fall within the equations mentioned under 206.

34. How should the subject of insurance be introduced to a class?

35. Explain the workings of a mutual insurance company.

36. A man has his house insured for $2000 in a mutual fire insurance company carrying a total insurance of $850,000. What will be his share in a fire loss of $1200?

37. Explain the various forms of property insurance mentioned.

38. The actual valuation of a dwelling house is $1400. What will be the premium at ½% per annum for 3 years? for 5 years?

39. A man insured his factory against fire for $50,000. This company reinsured for $15,000 in another company and $10,000 in a second company. What should each of the three companies pay in case of a fire loss of $1200?

40. Explain briefly the three forms of life insurance mentioned.

41. How should the subject of revenue be introduced?

42. Explain the various means of obtaining revenues.

43. What is meant by assessed valuation of property? How is it determined?

44. If the rate on a 60% valuation is .0341, what is the rate on the actual valuation?

45. The total property valuation in a school district is $220,000. How much must the tax be increased to improve the schoolhouse to the amount of $60 and to increase the teacher's yearly salary $90?

46. How much additional tax will a man living in the above mentioned district pay who owns property valued at $7500?

47. How much does a man owning $4500 of property pay towards maintaining the schools of his district if that tax is 2½ mills?

48. How and when are taxes collected in your state? What is the penalty for non-payment at this time?

49. Suggest six problems on revenue suitable for a grade class.

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CHAPTER XIX

BANKING AND INTEREST

210. General Banking. The bank in general and the savings bank in particular have replaced the stocking in the unused stove or the hole under the hearth as a depository of money. During the last twenty years the national bank deposits have increased from $1,715,194,859 to $6,611,-281,000. In view of this growing importance of the bank a thorough study of the elements of banking in the grades is imperative.

The primary functions of the modern bank are to receive deposits, to issue letters of credit for paying debts in other cities and countries, to loan money, and, in some instances, to change moneys from the denominations of one country into that of another. Banks performing some of these functions in one form or another have existed for over four thousand years. Any standard encyclopedia will give a good history of this important business institution. The following are the essential elements to be considered in banking:

1. Opening an account at a bank.
2. Making deposits.
3. The three forms of checks: payable to (a) John Doe, (b) John Doe, or order, (c) John Doe, or bearer.
4. The advantages of making payments by checks.
5. Stopping payment on lost checks.
6. Certification of checks.
7. The identification necessary for cashing checks.
8. Writing checks without sufficient funds to cover the amount of check.
9. Monthly statement.

211. Learning Directly About Banking. It will be well for the teacher to consult the local bank about any of the above matters concerning which he is not absolutely certain. A visit to the local bank by the class as well as a talk to the class by a bank employee will be a happy way of impressing the children with the realness of the bank as a business institution and of stimulating their interest in it as a branch of arithmetic. Excellent results have, also, been obtained by operating a school bank with play money.

212. Savings Banks. Savings banks usually pay from 3% to 4% interest, compounded semi-annually. The time used in computing interest on savings accounts varies in different states and communities. The following method is one of the most common: Deposits made after the first few days of the month—usually after the 5th day—draw no interest before the first of the following month. Interest is paid only on such money as is on deposit on the specified interest-paying date. In determining the amount of interest due each half-year all withdrawals are deducted from the sums longest on deposit. Suppose an account has $50 on deposit Jan. 1, that $60 more are deposited Mar. 1, then $40 withdrawn April 23, interest will be computed July 1 on $10, i. e., ($50-40) for 6 mo. and on $60 for 4 mo.; not on $50 for 6 mo., and $20 for 4 mo.

Ordinarily, savings banks pay out deposits on demand, but in most states they may require a 30- or 60-day notice. As a means of acquainting young people with the formalities and technicalities of banking, postal and school savings banks are being installed in many schools throughout the country. At the same time these innovations instill into the children at an early and impressionable age the habits of frugality and thrift.

213. Interest. Interest is a form of rent. A thorough treatment of the subject of interest should include:

1. A knowledge of the history of the custom of paying interest.
2. The need and justice of paying interest.
3. The maximum interest permitted in the state and the penalty for a higher rate.
4. The legal rate, or the rate charged in all cases of dispute where no rate is specified.
5. The making of the various forms of notes such as time notes, demand and coupon notes.
6. Securities in the form of real estate and chattel mortgages.
7. General local customs.

Much of the time usually spent in learning several methods of computing interest may be devoted with advantage to a consideration of the above items, which are such a vital part of the subject.

214. Interest as an Application of Percentage. Interest can well be introduced during the 6th grade as a general application of percentage through the equations

$$I = P \times R \times T \quad (1)$$
$$A = P + I \quad (2)$$
$$A = P \times (1 + R \times T) \quad (3)$$

The important thing and the only thing which may cause any difficulty in handling the above equations is that T must be expressed in years, or fractional parts of a year. Interest may be found on various principals at varying rates for:

1. An exact number of years.
2. A certain number of months, as, 6, 4, 3, 2, reducing them to the fractions of a year they represent; then, combination of months with years, as 2 years, 4 months.
3. Any number of months, as 2 years and 7 months.
4. Days, as, 10, 15, 20, giving simple fractions of a month.
5. Any number of years, months, and days.

There will be excellent opportunity in this connection for the children to use cancellation and they should be en-

Note:—As a precaution the student is warned against signing his real name to any practice papers.

couraged to do so. Problems for finding the principal, rate, or time may be treated most successfully as the reciprocal of finding the interest. Thus, the time during which \$1500 at 7% will give \$245 interest is indicated by the statement: $\$245 = \$1500 \times .07 \times T$, from which T may be found at once without difficulty. It is also a good plan to introduce these problems incidentally.

When the pupils have had considerable practice in finding interest, they may be required to find the interest on \$2500 for 1 year at 6%; this problem to be followed immediately by the question, What rate of interest will be necessary to realize \$210 interest per year on a principal of \$3000. From this on, *reciprocal problems* may be taken up shortly after the pupils are familiar with each of the various forms of time used in the direct problems.

215. Abbreviated Forms of Computing Interest. During the 7th or 8th grade, some abbreviated form of computing interest may be introduced. The following, which is based upon the fact that interest for 2 months at 6% is $\frac{1}{100}$ of the principal, is recommended;

Interest on \$1250 at 6% for 2 yr. 3 mo. 10 da. is as follows:

Interest	2 yr.	\$150.00
"	2 mo.	12.50
"	1 mo. ($\frac{1}{2}$ of 2 mo.)	6.25
"	10 days ($\frac{1}{3}$ of 1 mo.)	2.08
		\$170.83

At 7% interest $\frac{1}{6}$ would be added; at 5% interest $\frac{1}{6}$ would be subtracted, and so on. The tables in common use by banks should be explained at this time.

Until recently, exact interest has been used by the United States Government only, but it is being adopted more and more by the banks. In making a 90-day note they count

the exact number of days, not merely to the same date three months hence.

216. Compound Interest. The common use of compound interest in savings banks, and the enormous amounts of money thus loaned, make this an important item. For example, \$100 placed at 5% compound interest will amount to \$162.89 in 10 years, to \$338.63 in 25 years, to \$1146.70 in 50 years, and to \$13,149.24 in 100 years. The compound interest equation is

$$A = P[1 + r]^t.$$

217. Partial Payments. Partial payments is a form of interest problem which should receive only moderate attention; at the same time, it must not be ignored. However, it may wisely be restricted to two, or at most three payments in one problem. In computing such interest, the United States rule, which calls for exact computations, should be followed. Only a few short problems in partial payments need be solved.

218. Discounting Notes. As banks loan most of their money by discounting notes, this phase of banking should receive considerable attention. The case in which a man turns over to his bank a note made out to himself must be considered as well as that in which he borrows upon a note written by himself in favor of his bank.

Absolute accuracy is the only standard to be used in gauging any interest computation.

219. Exchange. In the early years of the study of arithmetic, much attention was given to exchange. This exchange was a complication arising from the multiplicity of coins and lack of uniformity in this coinage. Much of this material, which had a legitimate place in the arithmetics of fifty and a hundred years ago, has been retained until very recently. About all that needs to be included for

present needs is a comparison as to cost, safety, and expedience in sending money by registered letter, postal order, private check, draft, and telegraph.

Sufficient practice in converting United States money into English, French, and German money in foreign trade may be given the pupils to afford them a notion of international money exchange and equivalence in values. For example, English money is quoted as so many dollars and cents for one pound, as, $4.72. French and German money is quoted similarly, as so many cents per franc or mark, as 19c per franc or 24¼c per mark. It is, also, quoted as so many francs or marks for $1, as 5.21 francs, or 4.15 marks.

220. Banks as Collecting Agencies. Drawing upon a debtor in another city through a bank is an important business transaction. Mr. A. leaves with his bank a demand upon Mr. B. for payment; this is sent to a bank in the town where B. resides. The latter bank presents the demand to Mr. B. and if paid leaves him a receipt for the same remitting the sum to Mr. A.'s bank, less a few cents for collection. Travelers use this means of securing money by drawing upon their home bank through a bank in the city in which they may be stopping.

221. Travelers' Checks. A convenient way of carrying money on a journey is by travelers' checks sold by banks, express companies, and railroads. These consist of a small book of checks of varying amounts, each signed by an authorized person from whom they were bought and also by the holder of them. In obtaining the money represented by these checks as, at a bank or a hotel, where the holder may be stopping, he signs the check in the presence of the one paying it who compares this signature with the one given above.

XIX. EXERCISES

1. Discuss fully each function of banks mentioned under 210.

2. Give a short historical account of the banking institution.

3. Distinguish between private, state, national, and federal reserve banks.

4. Make a deposit slip; write one of each of the three forms of checks.

5. Mr. A made a check to the order of Mr. B, who lost it. A tramp found it, forged Mr. B's name to it and got it cashed by Mr. C. Whose loss was it? Why?

6. Discuss the nine elements of banking mentioned under 210.

7. Outline fully the management of a school bank.

8. What are the functions of a savings bank?

9. Find out from your local savings bank: (1) date of paying interest; (2) rates; (3) mode of computing interest.

10. Compute the interest at 4% due July 1 on the following savings account: deposits—Jan. 1, $40; Mar. 10, $15; June 1, $20. Withdrawals—May 2, $10.

11. Describe the postal and school savings bank.

12. Explain as to a grade class the meaning of interest.

13. Discuss, briefly, the history of interest; the reasonableness of charging interest.

14. What is meant by legal and maximum rates? What are they in your state? What is the penalty for charging above the maximum rate?

15. Write (1) a demand note; (2) a coupon note.

16. Outline the object of the work in interest during the 6th grade.

17. What is liable to cause difficulty at this time? Why? Illustrate. How can this be overcome?

18. Discuss the value of the interest equations. Illustrate.

19. What are the reciprocal problems in interest? Illustrate.

20. In what time will $2300 amount to $2500 at 6%?

21. Discuss the value of giving the class several different methods of computing interest. What short cuts are useful? When should these be used?

22. Find the interest, by the short cut given, of $2450 for 1 year, 5 months, 20 days, at 6%; at 7%; at 5%.

23. Find the interest of $1450 for 2 years, 7 months, 6 days, at 4%; at 7%; at 8% (by three abbreviated methods).

24. Find the interest on $1500 at 7% from Feb. 10, 1916, to July 15, 1916.

25. Find the exact interest for the above problem.

26. Make a table of the interest on $1000, $2000, up to $9000 at 6% for 1 mo., 2 mos., and so on up to 12 mos. Show the use of such an interest table.

27. A lot is bought for $600. If the tax is $12.50 per year, for what must it be sold at the end of 3 years in order that the investor may realize 6% interest on the investment?

28. A house rents for $30 per month, for which the taxes and upkeep are $160 per year. What could an investor afford to pay if money is worth 6%?

29. Solve problem 28, page 205, taking account of interest.

30. A merchant buys $1250 worth of goods for which the terms are 4 mo. or 5% for cash. What will he save by paying cash with money borrowed at the bank at 6%?

31. What is the legal value in your state of a 6 mo. note when due, if written for $2000 at 12%?

32. Find the compound interest on $250 for 4 years at 6%. Explain the compound interest equation.

33. State the United States rule regarding partial payments.

34. A note for $700 drawing 8% interest, made May 15, 1916, had the following payments made on it: Oct. 1, 1916, $200; Dec. 15, 1916, $400. How much was due May 15, 1917?

35. A note for $2500, dated Mar. 1, 1914, and drawing 6% interest, had the following payments indorsed upon it: $750 on Oct. 15, 1914; $350 on Dec. 1, 1914 and $600 on Mar. 1, 1915. What is the value of the note Oct. 1, 1915?

36. What is bank discount?

37. A note dated June 1, 1916, on $1200, drawing interest at 6%, due June 1, 1917, is discounted Oct. 1, 1916, at 8%. Find the proceeds.

38. A six-month note dated Nov. 1, 1916, on $2500, drawing 5% interest, is discounted Jan. 1, 1917, at 7%. Find the proceeds.

39. Mr. A discounts his 8% note on $1400, due in 3 mos., at the bank. Find the proceeds.

40. What must be the face of a 3 mos. note drawing 8% interest in order that the proceeds may be $750?

41. Find the date of maturity of loans made at a bank for 90 days on the following dates: Jan. 12, 1917; Feb. 23, 1916; July 5, 1917.

42. Explain the various ways of sending money, stating the cost, advantages, and disadvantages of each.

43. Explain the business relations between England and the United States if exchange on England is quoted at $4.75; at $4.95.

44. What will a £350 draft on London cost at $4.76?

45. What will a $1250 New York draft cost in London at $4.74?

46. What will a 2460 franc draft cost at 19½? at 5.15?

47. What will an $800 draft cost in Paris at 18¼? at 5.65?

48. What will a 3250 mark draft cost at 23¼? at 4.55?

49. What will a $750 draft cost in Berlin at 24½? at 4.16?

50. Explain the method of "drawing upon" a debtor in another city? Explain the process of drawing upon your own bank while traveling.

51. Explain "travelers' checks" and their advantages.

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CHAPTER XX

PARTNERSHIPS AND CORPORATIONS

223. Partnerships. Arithmetics published some years ago contained problems on partnership in which there were several partners, each of whom, from time to time, made a number of investments and several withdrawals all of different amounts. In all these problems strict account had to be taken of the time during which every investment of each partner was in the business. Such problems were so far removed from actual present-day practice that the whole topic has been almost abandoned in the later textbooks. Much of the business of the country is carried on in partnership and the topic, for this reason, deserves a prominent place in the arithmetics. As a rule, in such partnerships the withdrawals are so small that no account is taken of the time-element involved; often they take the form of regular salary paid each partner by the firm. If a large amount is withdrawn by one member for some length of time, a note is often given the firm by that member bearing the regular rate of interest, or some other similar arrangement is made. Arrangements will vary according to the risk of the capital invested, amount of capital invested by each partner, and estimated productive power of each member.

223. Treatment of the Interpretation of Statements. Any questions on interpretation of the statements in the problems which may arise should be settled at the time of its assignment. This question of interpretation, however,

should be a part of the pupil's work as far as possible since it is upon his ability to make such interpretation that his success in the solution of the problems will depend.

One illustration must suffice:

A earning $30 and B $18 per week, respectively, enter into a partnership, in which A invests $1000 and B $3000. How are the profits of $2100 for the year to be divided?

This division of profits will depend upon the agreement of the two partners, of which the following are possibilities:

1. The firm could give B a note of $2000 bearing current rate of interest; the balance, after this is paid, being divided in the ratio of $\frac{10}{18}$.

2. A might be so desirous of entering the business that he would be willing to divide the profits evenly.

3. The earning power of the money invested might be taken at current rate; and that of the men at the salary they drew when entering the partnership, all further profits to be divided evenly.

224. Corporations. Stocks and Bonds. Several criticisms of the arithmetic curriculum have advocated the elimination, from arithmetic study, of stocks and bonds. It is true that a radical modification of the topic, as it has hitherto been taught, is essential; but to abandon it completely should not be considered. Stocks and bonds have, for a long time, been popular investments for the masses in Europe and are daily proving more popular in this country. The number of stockholders in one of the largest railroad systems in the United States has increased 124% during the last twelve years. The universal $100 par value of shares in stocks and bonds of the arithmetics existed in the texts only. Several corporations are now, however, issuing stocks and bonds in small denominations of $500 and $100, called "baby" shares and "baby" bonds, for the accommodation of the many investors of limited means. Some corporations

are issuing shares of stock to the employees in order to secure their more hearty coöperation. In view of this growing popularity of stocks as a form of investment, it behooves us to give this topic a thorough common-sense treatment.

225. Stock Companies. Stock companies are often formed today to take the place of partnerships. In this way the number of partners can be made as large as desired and as a result, the sum invested can be increased to any desired amount. Changes in partnership, under such conditions, can, also, be made more easily. Each partner in the partnership is liable for any debt incurred by the firm, while the liability of a shareholder in a corporation is limited,—generally to the par value of the stock owned.

Local enterprises are often conducted as stock companies in order to arouse general interest in them. It is quite a common thing to increase the number of shareholders by making the par value of the shares low, sometimes only $25 or even $10, and then limiting the number of shares to be owned by any one individual.

226. Dramatizing a Stock Company. It is a good plan for the teacher to give the story of a supposed stock company to the pupils before they attempt the solution of problems under this head. This account should begin with the reason for organization and include an explanation of the following terms: Capital stock; par value; market value; relation between number of shares, par value and capitalization; sales of shares; dividends; preferred and common stock; "watered stock"; reasons for the variation of the market price from the par value; and any other interesting detail which may arise.

Taking a local stock company as a basis adds local color and interest. In the city such companies abound, but the few in the country as creameries, grain elevators, and stores

are better illustrations because of the wider familiarity with them on the part of the pupils. If no other illustration is available, the nearest railroad can be taken as an example. The story of the organization and direction of a corporation may also be developed in connection with the problems given for solution.

227. Bonds Defined. Bonds are merely notes given by a corporation instead of by an individual. As a rule, they are secured by some form of mortgage just as the ordinary note. They are bought and sold the same as shares of stock. Some items of interest should be studied at least briefly: the forms of bonds as government, municipal, and private corporations; reasons for issuing bonds; taxes on bonds; variation in interest; statements as K. C. Schools, 1919—$3\frac{1}{2}$ — 98; difference between stocks and bonds from the investor's standpoint, that bonds bring a constant income and hence fluctuate much less on the market; any other points of interest.

228. Cause of Child's Difficulty with Stocks and Bonds. The arithmetic involved in stocks and bonds is, in reality, very simple but is often looked upon as difficult by the average child because of a lack of comprehension of the principles involved and because of the mystery of the technical terms used. Cancelled bonds and shares of stocks may often be borrowed from a bank to show the class. This and a treatment as outlined above will clear up this matter and thus make the problems rational and easy of comprehension. Furthermore, a satisfactory treatment of the subject should, by all means, include these points.

229. Bankruptcy. When a firm or corporation is unable to pay its debts it is said to be insolvent and must turn over all property (assets) to pay the debts (liabilities), or to pay them as far as possible. This form of application of percentage is worthy of some study. This phase of

business should, at least, be placed before the pupils who should be required to learn the general terms.

XX. EXERCISES

1. Criticize the old form of partnership problems. What modifications are necessary to modernize this topic?

2. State several varieties of agreements under which partnerships may be formed.

3. Divide $2100 profits between A and B in the three ways named under 223.

4. Suggest three similar problems in partnership for a grade class.

5. Criticize the old form of teaching stocks and bonds.

6. Why should stocks and bonds be retained as a topic of arithmetic?

7. How many $100 shares must a stock company issue for a capitalization of $200,000? of $350,000? of $50,000?

8. What must be paid for five $1000 shares of stock quoted at 87½? What did they net the former owner? What was the brokerage?

9. What income do the stocks bring if this corporation declared a 2% dividend twice during the year? What rate of interest was this?

10. Compare the rate of income with that of a 6% farm mortgage?

11. What was the gain or loss, and rate of gain or loss, if the stocks were sold at the end of a year for 95? for 82½? for 76⅛?

12. Find the rate of gain or loss on stocks bought at 87½, when the uniform rate of interest is 6%, which declare a yearly dividend of 4%, if they are sold at the end of the year at 89½; at 82½; 84⅛.

13. Discuss the advantages of a corporation over a partnership.

14. Explain as you would to a class the organization of a stock company.

15. Explain the various terms used under 226.

16. Give some illustrations based upon local corporations.

17. Explain how the formation of a stock company can be shown in the organization of the school play bank.

18. Explain how the organization and direction of a corporation can be taken up in connection with the solution of problems.

19. Look up the daily stock quotations in your newspapers. Account for the great variations.

20. Suggest three problems from these quotations suitable for a grade class.

21. What are bonds? How do they differ from notes given by individuals?

22. Compare the rate of income from a 5% bond quoted at 92½ with a 5½% mortgage.

23. Why does the price of stocks fluctuate more than that of bonds?

24. Compare stocks with bonds as desirable investments.

25. Discuss the items regarding bonds mentioned under 227.

26. Consider Exs. 19 and 20 for bonds.

27. What are the difficulties encountered in the study of stocks and bonds? How can they be overcome?

28. What per cent of the indebtedness can a company pay whose assets are $56,340 and liabilities $83,565?

29. Explain the general idea of bankruptcy and give the meaning of the terms used in its connection.

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CHAPTER XXI

BILLS AND ACCOUNTS

230. Bills. Bills of various forms are met with very frequently by the average man and woman. This together with their simplicity makes them a convenient introduction to business forms, while the many number conditions which arise under them make bills a most fertile field for real practical problems. As soon as the children can write plainly, bills may be introduced as arithmetical exercise. This will be at least as early as the fourth grade. In playing at store or other business enterprises bills arise naturally. At first they would better be confined to two or three items in which addition is the only number work required. The number work is so increased by easy stages as to give practice in multiplication and addition of integers; multiplication and addition of fractions, including aliquot parts; changing United States to foreign money, and *vice versa;* computing discount; finding the amount of lumber, grain, and so on. Besides the general merchandise bill the following few special ones are recommended for study: Coal, gas, water, electric light, freight, and the monthly statement. Weekly work sheets and equation of accounts may be added in some instances. Much practice in filling out the various forms as well as in making the computations involved is advised.

231. Accounts. It is daily becoming more and more a necessity for every one to be able to keep simple accounts. The farmer, the small contractor, the produce man, the housewife, the mechanic, every man and woman in whatever

City of Emporia
WATER DEPARTMENT.
City Building, 20 East Fifth Ave. Office Hours: 8 A. M. to 5 P. M.
Emporia, Kansas, September 1, 1917.

Water Account for Month of August, 1917.

State of Meter..... Gl. Str. .....
Former Reading ..... Gl. No. .....
Consumed ..... Gl. $.....
Meter Rental - - - - - $.....

Total - $.....

Water Bills are due the **First of Each Month**; if not paid on or before the **Fifteenth of the Month**, water will be turned off and a charge of $1 extra will be made for turning on and off.

PLEASE BRING THIS NOTICE WITH YOU.

WATER

..... (NAME)
St.....
No.....
.....
.....
.....
.....

Form 12M-5-16-N15E
Electric Current Consumed, Month of _____
To The Kansas Electric Utilities Company, Dr.,
Bring This Card With You. It Will Be Your Receipt When Paid.

Book No.
Cust. No.

Present Reading	K. W.					
Former Reading	K. W.					
Amount Consumed	Rate					
	Discount 10%					
	NET					
	Balance					
	Total				Total	

NOTICE
Discount allowed only if paid on or before 10th. If not paid by the 15th the company reserves the right to discontinue service.

RATE:—Minimum Bill $1.00 per month. No Discount.
1 to 50 K. W. @ 10c per K. W.
51 to 100 K. W. @ 9c per K. W.
101 to 500 K. W. @ 8c per K. W.
501 and over @ 6c per K. W.
10% Discount if paid on or before 10th of month.

occupation engaged, needs to keep simple accounts in order to get the most out of his resources in money, labor, and energy expended.

A man who buys a house should keep an account with it in which he debits the purchase price, yearly insurance,

tax, repairs, and interest on the money invested, and credits it with the rent it would bring. He will thus have an estimate on the advisability of owning or renting a dwelling house and will be able to tell at any time the total amount of money he has invested. Farming can be done much more intelligently when a strict account is kept of the outlay and the income of each field and produce. Not a course in double entry bookkeeping or anything like it is here advocated but simple work in accounts which can readily be understood by the pupils of the last three or four grades.

Aside from the computation phase, the study of accounts will awaken the pupils to the importance of the economic issue involved. In addition to making out accounts in connection with the various business topics of arithmetic an account running for some little time may be kept by the pupils to be balanced once a week, as where the pupils play at keeping store or at being engaged in some other commercial enterprise. Above all, let the teacher encourage the pupils to keep accounts of their own incomes and expenditures, especially those who raise chickens, make gardens, sell papers, and so on. Statements of resources and liabilities given out from time to time by banks and various corporations are also suggested for study and application.

XXI. EXERCISES

1. When can bills be introduced into the arithmetic work?
2. What importance should be given to the use of bills in the grades? Why?
3. Give a list of the topics in which bills can be used to advantage and state what you expect to accomplish through their use in each case.
4. Suggest data for a bill of six items in which use is made of aliquot parts.
5. Explain the use of bills in one topic for giving reviews in

previously studied topics; for example, a lumber bill furnishing review in fractions. Give additional illustrations.

6. Explain the electric light bill given on p. 223. How much does Thomas waste if he leaves two 25 W lamps burning over night? What is the cost per hour of using a 577 W electric iron?

7. Write a bill for (a) electric light, (b) gas, (c) water, (d) a load of coal.

8. Make a monthly statement and explain the same.

9. Select from some newspaper a business report of resources and liabilities and explain the same.

10. Explain the weekly work sheet and equation of accounts.

11. What is meant by "debit" and "credit"? Explain the form of account which should be taught in the grades. What should be the main object?

Make a simple account for each of the following three statements:

12. Bought a house Dec. 1, 1915, for $3500, which was leased for one year for $28 per month. Paid the following during the year: Jan. 10, $38 for three years' insurance; Feb. 1, $76.40 tax; April 10, $18.75 papering and painting; Sept. 15, $12 repairing roof; Nov. 1, $4.50 repairing furnace. Was this a profitable investment?

13. A farmer planted 60 acres in corn, which produced 53 bushels per acre worth 42 cents per bushel. The farm was valued at $55 per acre and the owner paid a yearly tax of $82 on a quarter section. The cost of production was seed $24, fertilizer $65, 53 days' work by two teams at $2.50 for each man and team.

14. As an experiment, corn was planted upon 20 acres of the same farm which was not fertilized. The yield was 31 bushels per acre and the cost $8 for seed and 15 days' work for two teams. Compare the cost per acre, per bushel and rate of return upon the capital invested in problems 13 and 14.

15. Fill in the blanks for the following records of a day's business:

	Cash on hand	Sales	Disbursements	Balance
Monday	$256.40	$ 96.34	$ 21.40	.....
Tuesday	331.34	86.30	.....	$125.60
Wednesday	125.60	.....	115.34	104.36
Thursday	104.36	109.86	.....	95.24
Friday	95.24	76.34	92.00	.....
Saturday	.....	178.35	68.40	189.75

16. Suggest data for a problem on gardening and selling its produce. Is this suitable for school gardening?

17. Suggest data on the raising of chickens.

18. Give other child activities suitable for keeping accounts.

19. Explain the value of keeping such accounts from both the arithmetical and economic standpoints.

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CHAPTER XXII

LITERAL ARITHMETIC

232. The Literal Equation. The *literal equation* is a kind of shorthand mathematical statement. Thus, the statement that the total area of the surface of a cone equals the sum of the product of *pi*, (π), radius, and slant height, plus *pi* times the radius squared, may be written briefly:

$$S = \pi rs + \pi r^2.$$

All formulas, and, indeed, the majority of arithmetical statements can be expressed with greater brevity and also with much greater clearness in this shorthand language. Further illustrations will be found in this chapter, in the various topics of arithmetic, and in magazine articles discussing quantitative relations.

233. Reason for Teaching the Literal Equation. An important reason for teaching the arithmetical equation is that many pupils leave the elementary schools for occupations in which they will find it greatly to their advantage to be able to read intelligently technical articles regarding their particular line of work. A few illustrations will be found under **242.**

234. A Duty of the Teacher. In order that this work may be made effective it is necessary for the teacher to recognize that these so-called shorthand equations are a *part of the arithmetical language;* not *algebra.* This work is not to serve as a forerunner of high school algebra, but rather as a substitute for algebra for the pupils who can

not later enjoy that study. For this reason the teacher is cautioned about the use of such expressions as "coefficient," "transpose," "clear of fractions," and other algebraic terms likely to prove confusing to the student thus obscuring an otherwise simple subject.

235. Introducing the Literal Equation. Introduce the literal equation as a shorthand expression of arithmetical statements. The very reference to shorthand will in itself prove a stimulus. There is scarcely a child but has an ambition to know shorthand. This is used by grown-ups. A child is first asked to write upon the board the law for subtraction:

Minuend minus subtrahend equals the remainder.

Let the teacher now explain that there is a shorthand way of writing this expression in which signs and only the first letters of the words are used. When the proper symbols for the words have been agreed upon after a discussion by the pupils, they are written immediately below the word they represent. Thus the above would be:

Minuend	*minus*	*subtrahend*	*equals*	*the remainder,*
M	—	*S*	=	*R*

This introduction may properly be made as soon as the children can read problems and write sentences like the above. This may be as early as in the fourth grade, or may be delayed to the sixth or seventh grade. When the literal equation has once been begun, however, *it must be used continuously* or the time already consumed in learning it will be wasted.

236. Problems as Material for Translating. Problems furnish excellent material for this work of translation. No attempt need be made at first in using the equation for the solution of problems. The attention should be placed wholly upon recasting the statement into this new language-form.

Extreme care is necessary, too, in the selection of problems for this early translation work. Facility and accuracy in making this translation are the necessary foundation for the satisfactory use of the literal equation in the solution of problems. The children should have early practice in the translation of words into symbols before they attempt any solution of problems. Teachers often fail in making this shorthand language effective because they present the main, general ideas one day and expect their pupils to be able to apply them to the solution of problems the next.

237. Treatment in Later Grades. Teachers of any of the higher grades whose pupils come to them unacquainted with the literal equation should introduce it at once. Although faster progress can be made with these advanced pupils, it will still be necessary for the introduction to be simple and for the work to be commenced some weeks prior to the time it will be needed in the solution of problems.

238. Literal Equations in the Solution of Problems. A most important application of the literal equation is the solution of inverse problems, as may be seen in the following examples. For instance, to solve any of the three inverse interest problems it is only necessary to substitute the given quantities in the general equation

$$I = P \times T \times \frac{R}{100},$$

thus reducing the solution to a simple division. This fundamental equation is easily remembered and recognized. One perfectly clear statement thus takes the place of several stereotyped, vague rules which are hard both to remember and to discriminate between because of their close similarity. Mathematically they are the same—a fact which should be

emphasized, not covered up. From the standpoint of simplicity and clearness the literal equation has every thing in its favor over innumerable rules for the solution of such inverse problems. The fundamental equation of percentage,

$$P = B \times R;$$

furnishes a means for handling the problems of percentage far more efficiently than can be done by a number of rules.

239. The Literal Equation with Ratio and Proportion. Still another topic which is very greatly simplified by the use of the literal equation is ratio and proportion. Indeed, wherever it is used the literal equation is one of the greatest instruments for unifying and rationalizing arithmetic, thus paving the way to a simpler, more intelligible, and more usable arithmetic.

240. The Solution of Problems with Literal Equations. When the pupils have a clear understanding of the language of the literal equation so that they can substitute it readily for its English equivalent they are ready for the solution of simple problems. It is wise to select such, at first, as give integral solutions. For example:

"A man buys a lot for a certain sum and builds a house upon it costing five times as much as the lot. Together they cost $3600. What is the cost of each?"

The following are suggested forms for the equations:

$$\begin{aligned} & CL + 5CL = \$3600 \\ \text{hence} \quad & 6CL = 3600 \\ \text{and} \quad & CL = 600 \\ \text{and} \quad & 5CL = 3000 \end{aligned}$$

When these are thoroughly mastered the pupils who have had work in fractions may take up problems of the following type:

"Find the cost of a horse sold for $150, an advance of one-tenth of the cost price."

The equation here will be:

$$C + \tfrac{1}{10}C = \$150$$

hence $\tfrac{11}{10}C = 150$

and $C = 150 \times \tfrac{10}{11}$

or $C = 136.36$

241. Advantages of the Equation Method. The idea of reciprocal processes should be used in finding the value of the unknown from the second equation, finally reducing the problem to that of finding the number which, multiplied by a given number, will produce a known product. The equation connects the numbers involved and also suggests the operation necessary to find the value of the unknown quantity. The number relations are thus made clear to the eye and no guess work is needed to decide what operations are necessary, what numbers are to be selected for the various operations in solving the problem.

242. The Fundamental Relations Translated into Sign Language. All the fundamental relations appearing from time to time in the study of arithmetic, such as the connection between the numbers in multiplication and division, areas of surfaces, and so on, should be translated into this sign language. The application of each to the solution of problems is, of course, to be included throughout. In the advanced classes it may be a good plan to introduce for interpretation and for finding values of any quantity in terms of other given ones, some of the equations appearing in technical journals. The following illustrations must suffice:

1. The friction of ropes over pulleys due to their stiffness is given by the equation

$$S = k \frac{d^2}{r} T,$$

d = diameter of the rope
r = radius of pulley
T = tension on the rope
k = constant for the particular rope tested.

2. A common formula for the length of cross belts is

$$L = 3\tfrac{1}{2} (R + r) + 2 \times l,$$

R = radius of one pulley in inches
r = radius of other pulley in inches
l = length between centers of pulleys in feet.

3. Automobilists often use the following formula to compute the horse power of their machines:

$$H.P. = \frac{(D + L)^2 N}{9.92}$$

D = diameter of cylinders in inches
L = length of cylinder stroke
N = number of cylinders.

4. Foresters use the following formula to give the number of board feet in a log:

$$Bf. = \frac{(d - 4)^2 L}{16}$$

d = diameter of log at smaller end in inches
L = length of log in feet.

The teacher must be careful in giving this work not to assign impossible or improbable values to the quantities in any particular formula. Such errors have the tendency to make the work bewildering and will detract from the pupils' interest and dependence on it.

243. Problems Without Numbers. Problems without numbers is another form of literal arithmetic. These problems are in the nature of questions in which the pupils

are asked to state how they would go about solving problems with certain given conditions, as,

"How can the price of several articles be found if the price of one and the number of articles are given?"

Many of the present day texts contain lists of such problems but unfortunately they are often placed in one group at the end of the book and hence most teachers regard them as an addendum of secondary importance, which is usually "not reached" at all. These problems are of anything but minor importance and should not be overlooked. Their daily use throughout the grades will produce excellent results but their value when taken allopathically at the close of the subject is exceedingly doubtful. Problems without numbers can not be solved by trying various number combinations until the answer in the back of the text is secured. In order to answer the questions the child must actually know the basic idea back of the solution and his very answers will aid him materially in securing a firm grasp of these methods of solution.

244. General Must Be Followed by Specific Questions. Unless these general questions are followed by questions of a specific nature this work will, of course, be of little value. The answer to the question, "How would you prove an example in division," is at once applied to several such examples as:

"Have I performed a division correctly if I get a quotient of 56 and a remainder of 17 in dividing 11,941 by 213?"

Again, in order to make this work effective it is necessary for the teacher to require at all times definite statements in clear English sentences. Such drill will be found excellent practice in the formulation and expression of ideas, an art in which the average pupil is sadly deficient.

245. Illustrations. As illustrations of the kind of problems suitable for use in the various topics, the following may be mentioned in connection with the subtraction of integers:

1. If the sum of two numbers and one of them are known, how can the other be found?
2. How can you find the cost of a house when the cost of the lot and house together, and that of the lot are known?
3. How can the number of apples John has be found, after he has sold 5?
4. John returns from town with 20 cents. If you were told how many cents he had when he left home, how could you find how many he spent?
5. John has 20 cents more than James. How can you tell how many cents John has if you know how many James has? How can you tell how many cents James has if you know how many John has?

Teachers will find little difficulty in suggesting similar lists of problems in the other lines of arithmetical work and after a little practice will be able to dictate them extemporaneously to the class. The only caution to bring to the attention of the teacher is, that this excellent aid to the teaching of arithmetic can be neglected only at the risk of robbing the pupils of something that will be of inestimable benefit to them, not only in their arithmetic, but in developing their reasoning powers as well.

XXII. EXERCISES

Give symbolical expressions for the following eight statements:

1. The area of the entire surface of a rectangular parallelopiped.
2. The area of the entire surface of a pyramid of square base, whose sides are L and slant height s.
3. The volume of the above pyramid.
4. The area of the entire surface of a regular pyramid of square base $2L$ and altitude h.
5. The area of the entire surface of a right circular cylinder.

6. The area of the entire surface of a right circular cone.

7. The rate of gain (loss) in terms of the buying and selling price.

8. Amount from principal, rate, and time of money loaned.

9. What is the equation connecting dividend, divisor, quotient, and remainder?

10. Show that all even numbers are represented by $2n$ and all odd numbers by $2n$-1, where n is an integer.

11. Distinguish between literal arithmetic and algebra.

12. Explain how "literal arithmetic is a sort of shorthand language."

13. Illustrate Ex. 12 by several arithmetical laws.

14. Suggest and solve six problems applicable to Ex. 13.

15. What are inverse problems? Illustrate.

16. Explain the use of the literal equation in their solution.

17. Explain the advantage of the literal equation over the usual set of rules.

18. State any further advantages favoring the use of the literal equation.

19. When should work in literal arithmetic be begun?

20. Give a first illustrative lesson.

21. Give a short development of the use of literal arithmetic throughout the study.

22. What is meant by "Problems without numbers"? Illustrate.

23. How can the gain or loss be found from the cost and selling price of an article?

24. How can the cost of an article be found when the selling price and the gain or loss of the sale is given?

25. How can the selling price be found when the cost and the gain or loss per cent is given?

26. How can the gain or loss per cent be found when the cost and selling price are known?

27. How can the price of any number of articles be found if the price of any other number is given?

28. The weight of a wagon full of wheat and the empty wagon are given. How can the number of bushels of wheat be found?

29. The weight of a tank full of water and the empty tank are given. How can the number of gallons the tank contains be found?

30. Given the gain or loss per cent and the selling price, how can the buying price be found?

31. Given the marked price and the discount of an article, how can the cost be found?

32. Give an outline of the work in problems without numbers to be used throughout the study of arithmetic.

33. Illustrate the use of these problems in connection with any regular lesson in the arithmetic.

34. Consider Ex. 32 in greater detail for any one topic.

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CHAPTER XXIII

GRAPHS

246. Why Taught. Picture schemes and graphs show contrasts and relations much more clearly and vividly than do columns of figures. The relative military strength of nations is vividly portrayed by soldiers of varying sizes; the productiveness in wheat and in other grains of different countries by sacks or stacks of grain; the value of bank deposits from year to year by bags of money; and so on. Maps drawn to scale show the relative sizes of towns, states, or countries; an architect's plan explains the relative size of rooms. This most effective instrument for bringing quickly to the reader a clear idea of variations and comparisons is now used universally by magazines, newspapers, Experiment Station and Government publications.

As it is also a subject which comes well within the mental range of the children of the upper grades and one which can be made exceedingly interesting to them, its introduction into the work of these grades is strongly recommended, even though it has not, as yet, found a place in the average textbook. The time which can be saved and the difficulties which can be avoided in the final consideration of ratio and proportion will alone justify the prior study of graphs. Teachers of graphs in the grades must ever bear in mind the fact that this work is not to be treated as a forerunner of algebra for those who will later go to high school, but as a *final subject* for those who will not later have the advantages of high school study.

247. Introducing the Subject. Four or five pictures showing comparisons, as those mentioned above, cut from magazines and newspapers, and mounted upon cardboard will be found good material for introducing the subject. After the use of such comparative pictures has been discussed freely by the children, suggest that they be on the lookout for other illustrations in the magazines and newspapers and that each bring to school within a week, at least one mounted illustration. When these have been brought in, the study may be continued for another lesson, or several if necessary, during which time the pictures may be looked over and each explained by the pupil who brought it. Let this material now be pinned upon the wall or placed in some convenient place for future study. Some of the more promising pictures may be retained by the teacher for use with later classes.

248. Picture Versus Continuous-Line Graphs. The distinction between picture graphs for comparisons of isolated cases, and of the continuous-line graph must be made clear. The graph of the motion of a football during a game, as given in exercises 2 and 3, listed on page 240, will be found excellent material to create interest and to give the idea of a continuous change.

249. Locating Points. Latitude and longitude furnish a simple illustrative example of coördinate axes and their use in locating points. Most cities number the houses north and south, and east and west from two streets intersecting at right angles. Where this is the case, the teacher has a ready-made local example of coördinate axes. With these streets as reference lines let a plot of the city now be made by the pupils and points of interest located on it, such as the schoolhouse, post office, library, and even the homes of the pupils, unless some local condition should make this inadvisable.

In rural communities, range and township lines can be used similarly. All such maps should be retained for future drill upon the idea of positive and negative direction. The coördinate axes may also be drawn and explained arbitrarily. A careful study of the order of the appended exercises is recommended as a guide in carrying out the subject in the grades.

250. Essentials of Graphs. A knowledge of graphs requires ability to read and interpret given graphs, as well as their construction from given data. This must embrace an understanding of the following essentials:

1. Coördinate axes and their use in locating points.
2. Positive and negative directions.
3. Units used.
4. Use of the coördinate axes in expressing conditions existing between two quantities.
5. Expression of a continuous relation existing between varying conditions by means of a curve.
6. Placing of the independent measure along the horizontal axis, thus making the readings sought along the vertical axis.
7. Special points in a graph and their significance.
8. Interpretation of conditions not shown in the graph.

251. Material for Problems. A few easily available sources of material exercise in graphing are here suggested:

1. Temperature curves for varying periods.
2. Interest tables for varying rates.
3. Price of a number of articles from the price of one.
4. Number of bushels of grain in varying net weights.
5. Fluctuation of prices of grain, stock, etc.
6. Attendance and tardy record of the room or school.
7. Statistics found in year books and yearly newspaper almanacs.

XXIII. EXERCISES

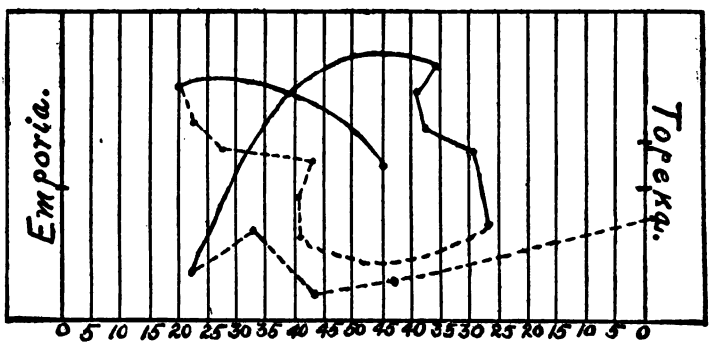
1. Mount on cardboard for future use three picture forms from magazines or newspapers, which show the contrast sought more clearly than columns of figures would do. Explain each as to a grade class.

2. Answer the following questions for the accompanying graph.

i. How far was the ball punted toward the goal of Topeka High School? toward the goal of Emporia High School?

ii. What was the actual distance that the ball was kicked in each of the above two instances?

iii. The distance kicked toward the goal was what per cent of the total distance kicked in each of the above two instances?


Emporia.
Topeka.
0 5 10 15 20 25 30 35 40 45 50 45 40 35 30 25 20 15 10 5 0
Emporia's punt to Topeka.
Emporia's ball.
Topeka's punt to Emporia.
Topeka's ball.

iv. How far was the ball carried toward the Topeka goal? toward the Emporia goal?

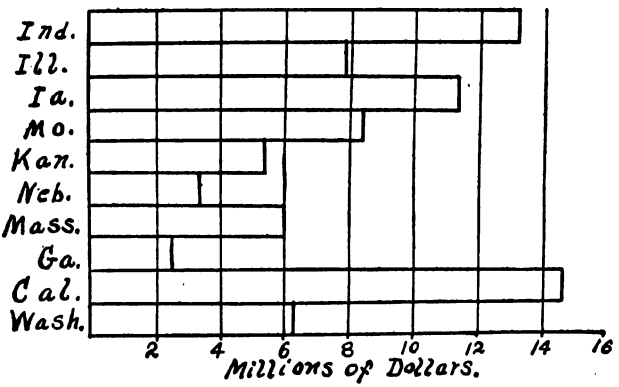
v. What was the total distance that the ball was carried by Topeka? by Emporia?

vi. What per cent of the total distance that Topeka carried the ball was toward the Emporia goal? that Emporia carried the ball toward the Topeka goal?

3. Draw a football field to scale and show the following plays: Emporia "kicked off to" Topeka and the ball was caught on the 20-yard line 15 yards from the left side line. The Topeka player who caught the ball returned it 10 yards directly toward the Emporia

goal before "down." First "down," 4 yards forward and 10 to right. Second "down," 2 yards forward and 12 to right. Third "down," 2 yards forward and out of bounds to the right. Fourth "down," "punt" 40 yards forward and 20 from the left side line. The Emporia player catching the ball brought it directly forward 15 yards. First "down," 12 yards forward and 8 to the right. First "down," 5 yards forward and 10 to the left. Second "down," 7 yards forward and 5 to the right. First "down," end run past goal line with the ball "down" 15 yards from the right side line.

4. The following will be referred to as a "picture graph" to distinguish it from graphs proper. Picture graphs will be found



Graph showing expenditures on good roads by ten states.

useful in showing conditions which do not depend upon continuous changes.

Explain the graph and its object fully.

5. Make a picture graph for the following data on the weight of an average man 5 ft. 5 in. tall for different ages.

Age	Pounds	Age	Pounds
15–24	134	45–49	147
25–29	138	50–54	149
30–34	141	55–59	149
35–39	143	60–64	148
40–44	146	65–69	147

6. Make a picture graph from the following data of the illiteracy of the United States in geographical divisions. Explain.

United States	7.7 per 1000
New England	5.3 " "
Middle Atlantic	5.7 " "
E. North Central	3.4 " "
W. North Central	2.9 " "
South Atlantic	16.0 " "
E. South Central	17.4 " "
W. South Central	13.2 " "
Mountain	6.9 " "
Pacific	6.0 " "

7. Make a picture graph of the following data of the rainfall in 1914 at Emporia, Kansas.

January ..... .03	May 2.14	September....2.18
February.....1.70	June.....1.97	October 2.32
March 2.71	July 3.88	November .... .09
April.....1.74	August...3.58	December..... .77

8. Draw two axes intersecting at right angles near the lower left hand corner of the page each eight inches long. Label the right extremity of the horizontal axis, "Depth in ft." and the top of the vertical axis, "Pressure in lb." Mark each inch along the horizontal 5 and along the vertical 2. Locate each point upon the diagram represented by the following table giving the readings for pressure of water at different depths:

Depth in feet.	Pressure in pounds.
6	2.60
8	3.40
10	4.33
15	6.49
20	8.66
25	10.82
30	12.99
35	15.16

Beginning with the highest point in the diagram, draw a smooth curve through the various points. What use can be made of this graph? What is the pressure at a depth of 12 ft.? of 27 ft.?

At what depth is the pressure 7 lb.? 13.5 lb.? Could the graph be extended so that it would give the pressure at 45 ft.?

9. Graph the following data on the population of Kansas:

1860	107,206
1870	364,399
1880	996,096
1890	1,427,096
1900	1,470,495

What was the approximate population in 1875 and in 1885?

10. The following table was prepared at the United States Weather Bureau showing the relative humidity based upon observations taken at the same hour daily for fourteen years. Plot the two sets of observations to the same axes and state the points of interest which they show.

	Jan.	Feb.	Mar.	Apr.	May	June
Dodge City, Kan.	72%	72%	64%	61%	64%	63%
Santa Fe, N. Mex.	55%	55%	43%	35%	36%	31%

	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Dodge City, Kan.	62%	62%	63%	64%	66%	69%
Santa Fe, N. Mex.	47%	47%	46%	47%	48%	55%

11. Graph the following table of parcel post rates using a different curve for each zone. What practical use can be made of this?

Lbs.	Zone 1–2 0 to 150 mi.	Zone 3 150–300 mi.	Zone 4 300–600 mi.	Zone 5 600–1000 mi.	Zone 6 1000–1400 mi.	Zone 7 1400–1800 mi.	Zone 8 Over 1800 mi.
1	$0.05	$0.06	$0.07	$0.08	$0.09	$0.11	$0.12
2	.06	.08	.11	.14	.17	.21	.24
4	.08	.12	.19	.26	.33	.41	.48
5	.09	.14	.23	.32	.41	.51	.60
10	.14	.24	.43	.62	.81	1.01	1.20
15	.19	.34	.63	.92	1.21	1.51	1.80
20	.24	.44	.83	1.22	1.61	2.01	2.40

12. Make a graph for any railroad time card.

13. Make a graph of two time tables of trains going through a division point. These should show trains going in both directions.

How can these graphs be used to show what connections can be made at this division point?

14. What is the present status of graphs in arithmetics? Discuss the advisability of a change.

15. Explain upon the board as to a grade class the idea of co-ördinate axes using latitude and longitude or two streets as illustration.

16. Suggest an introductory lesson on graphs as to a class.

17. Select data for three problems suitable for picture graphs.

18. Graph the data for any one problem in 17 and point out the important features which the graph portrays.

19. Select data for three problems suitable for graphing by grade pupils.

20. Graph the data of any one problem in 19 and explain the important features which it portrays.

21. Why is it not proper to connect the extremities of the rectangles in graph II or of the picture graphs made from data in problems 5-7 as was done in graphing data of problems 8 and 9?

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CHAPTER XXIV

RATIO AND PROPORTION

252. History. Our theory of ratio and proportion is an outgrowth of the *Rule of Three,* also referred to as the *Golden Rule,* because of its great power in the solution of problems. The Rule of Three dates back as far as the Hindus and Arabs, and held a very prominent place in the arithmetics of Europe and America until about a century ago. Besides the *Inverse Rule of Three,* an extension, similar to our compound proportion and called the *Chain Rule,* was formulated. All of these were rules in the strictest sense of the word, and were followed arbitrarily by the computer.

The value of these rules lay in the fact that they could be used mechanically in the solution of problems without any thinking on the part of the computer. They were machines, with whose mechanism the user was wholly unfamiliar. During the last century these rules have largely been replaced by *analysis.* It should also be remarked that the long, complicated problems in compound proportion, on digging ditches and the pasturing of cattle, so prevalent a few decades ago, have given way to problems more nearly probable.

253. Change of Notation. A most interesting phase in the history of *Ratio and Proportion* is the gradual change in the notation used. Some of the more noticeable forms were:

4 marbles || 8 cents || 9 marbles (1)
4 marbles — 8 cents — 9 marbles (2)
4 marbles . 8 cents :: 9 marbles (3)

Because of the use of the point in decimals and as a sign of multiplication, the latter was changed to,

4 marbles .. 8 cents :: 9 marbles, (4)

which has evolved into the form still prevalent in our texts,

4 marbles : 9 marbles :: 8 cents : ? (5)

It should be noticed that the fourth term, or the question mark representing it, is absent in the early forms. The early writers of arithmetic did not appreciate the real idea of ratio, or that a proportion is an equality of ratios. For this reason, the symbols used served only to separate one number or group of numbers from the others and were selected without any reference to their meaning.

The last form (5) has largely been changed to the more rational one of

4 marbles ∶ 9 marbles = 8 cents : ? cents (6)

but so far the one meaningful form

$$\frac{4 \text{ marbles}}{9 \text{ marbles}} = \frac{8 \text{ cents}}{C \text{ cents}} \qquad (7)$$

has not yet received universal recognition.

254. Ratio. Ratios arise continually in every day affairs and their nomenclature contains some of the most common mathematical terms including such familiar expressions as "they are in the ratio of," "they are proportional to," and so on. Ratio asks: *How many times one number or quantity is that of another.* As illustrations, one may cite comparisons in commercial activities,

such as the cost of living now compared with that of ten years ago; quantitative comparisons of physical changes, as the variation in volume of a gas when pressure is exerted on it.

All measurements are examples in ratio. Thus, the jeweler uses the term *carat* to designate the ratio in twenty-fourths of fine metal in a piece of jewelry; for example, 17 carats gold is $\frac{17}{24}$ fine gold. The so-called converting factors, by means of which one system of measures is changed into that of another, as for instance, the 39.37 used to convert lengths from the metric system into those of the English, and *vice versa*, are also examples in ratio.

Any comprehension of the number relations of arithmetic requires an understanding of ratio and its terminology. It is not necessary to devote a separate topic to this subject in the text; nevertheless, it must not receive the neglect that many of the children of the public schools show it is receiving at present.

255. Common Definition Source of Much Confusion. The difficulties experienced by children in the study of ratio originate in the very first lesson. Most texts define ratio as the relation between two numbers. *What* this relation is, is never specified. This definition is followed by the illustration 3 : 4 which the child might well consider equivalent to 10 : 11, since the first term in each is 1 less than the second. The fact that ratio is concerned with the number of times one number goes into another, that it is a question of division, must be brought home clearly to the children by the teacher. In order to make this point clear, much attention especially at first must be given to illustrations. The unfortunate notation 3 : 4 has done much to cover up the fact that ratio is a *comparison by division*, since the notation: used by Leibnitz for division, is not now so used in the United States. Hence, if this meaningless notation is

used in the text, it should be discarded for $\frac{3}{4}$. In order to substitute the notation $\frac{3}{4}$, it is suggested that the work on ratio be first taken up *outside* of the text in which this notation is used and the children be required, thereafter, to use it regardless of the notation found in the text.

256. Proportion. The teaching of proportion has been even more greatly handicapped than that of ratio as a result of a meaningless notation. As a rule, the present day texts correctly define proportion as the equality of two ratios, at the same time that they use the meaningless notation

$$3 : 4 :: 7 : ? \qquad (8)$$

in the place of the rational one

$$\frac{3}{4} = \frac{7}{N} \qquad (9)$$

Children who have been taught the proper meaning of ratio, together with its correct notation, $\frac{3}{4}$, will readily accept the last correct form for proportion and supplant any other form in the text with it. Through comparison, which is the fundamental idea of proportion, an equation is built up involving the three given quantities and the fourth one to be found, thus reducing the problem to the solution of a literal equation.

257. Omit Complicated Proportion Problems. The long, complicated problems involving unlikely conditions, as mentioned above, have done much to place the subject of proportion in disrepute. Omission of such problems is most commendable; not, however, the subject *in toto*, as will be discussed later. The highly complicated problems which are often offered in the text must either be simplified by the teacher or others which are within the range of the understanding and experience of the pupils should be substituted.

Strictly speaking, ratio is the comparison by division of two quantities having the same unit of measure. It is hence best to introduce proportion by the use of the form:

$$\frac{9 \text{ marbles}}{4 \text{ marbles}} = \frac{C \text{ cents}}{8 \text{ cents}} \qquad (10)$$

rather than by

$$\frac{9 \text{ marbles}}{C \text{ cents}} = \frac{4 \text{ marbles}}{8 \text{ cents}} \qquad (11)$$

As long as form (10) makes the work clearer that is the form to use but as soon as a child notices that

$$\frac{9}{C} = \frac{4}{8} \qquad (12)$$

expresses an absolutely correct number relation he should be permitted to use it. A too strict adherence to the use of any form as (10) above will make the child feel that the teacher is carrying a hobby too far. The child is right. It is *number relations* which are being dealt with and the relation (12) is perfectly true. Here lies one of the teacher's greatest opportunities for rationalizing the arithmetic work in general, and that of proportion in particular, so as to make the children feel that it is based upon a real common-sense basis.

258. Proportion Treated by the Equation Rather Than by Unitary Analysis. As long as proportion meant the solution of long, complicated problems by arbitrary rules, or by the use of a meaningless notation which embodied also an arbitrary procedure in the setting up of forms, it is not surprising that rational teachers have been discarding proportion for unitary analysis. A rational application

of the subject through a meaningful notation as herein suggested, is really an abbreviated form of solution by unitary analysis and possesses too many advantages to be wholly discarded.[1] In the first place, proportion is an extension of the number relation found in ratio, the value of which has already been discussed. The idea of proportionality so often referred to in business and every day affairs can, thus, be made clear to the pupils. By careful teaching, the shorthand equation form of solution can be made as clear to the pupils as the longer form of so-called unitary analysis. This may be brought about by the repeated solution of many problems, at first using the simplest data in which every child is required to be ready to state his reason for every ratio and proportion which he uses.

259. Graphing Direct and Inverse Proportions. The graphing of relations which are directly and inversely proportional, as well as some which do not possess either of these relations, will be found a great help in teaching proportion. The graph of direct proportionality, by far the most important case, will soon be recognized by the children and from it the idea of direct proportionality. The literal equation affords excellent opportunity for a hasty inspection of the form of solution in which unitary analysis does not give practice.

260. Solving Problems by Proportion. In solving problems by proportion, the pupil should give his whole attention first, merely to setting up the equation; later his undivided attention may be given to the mechanical process of solving the equation for the unknown number. This power of concentration is, in itself, an advantage in the solution of problems and a quality to be fostered.

261. A Plea for the Reasonable Use of Proportion. A problem may often be shortened by opportunities to cancel or to approximate results from the statement of the equa-

tion, which do not present themselves so strikingly in solutions by analysis. This is not a plea for the exclusive use of proportion in the solution of problems. It is rather a plea for the retention of a most serviceable arithmetical instrument which has fallen into disrepute by reason of its having been applied to needlessly long complicated problems, and by the arbitrary application to it of an irrational and largely meaningless notation. Unitary analysis, or any other form of solution is suggested for any case in which such a solution affords a clearer explanation or a more expeditious solution than proportion.

262. Ratios that Seem Equal, Not Necessarily So. It is the teacher's duty, in connection with ratio and proportion, to call attention to a matter which is often omitted from the texts; namely, that ratios which at first seem to be equal are not always so. In the purchase of 2 hats and of 200 hats the ratio of the costs would not be $\frac{1}{100}$ because the 200 hats would be bought at wholesale for a lower cost than the 2 hats would be sold for at retail. This is an important consideration which often arises in business.

The graph is again suggested as a means of clearing up any questions arising as to the direct, or indirect, proportionality between two quantities.

263. Majorities, Pluralities, Averages. Various particular forms of comparisons of numbers exist in our modern business and political life. Among these come first the ones in which the excess of one number over one or more others is found as in the case of majorities and pluralities. *Average* is a simple case of ratio. The average weight of the men on a football team, or the average salary paid teachers of a state are good illustrations. Another illustration is that of a baseball player's fielding and batting average which is usually expressed as a three-place decimal, but often erroneously spoken of as percentage. Yet another form which is

quite similar to the last one mentioned is that of stating the numbers out of each 100, 1000 or 10,000, as death rate which is given as so many per 1000 persons. The common use of these terms makes it well worth while to include them in the arithmetic work.

XXIV. EXERCISES

1. What is the Rule of Three—direct and indirect?

2. Explain clearly what is meant by ratio. Illustrate. How closely should this definition be adhered to?

3. Discuss the definitions and notation of ratio found in the average texts.

4. What is the ratio of 3 in. to 1 ft.? 4 in. to a yd.? 6 in. to 2 ft.? 3 yd. to 2 m.? 2 pk. to 1 bu.? 6 pk. to 2 bu.?

5. What is the ratio of the circumference of a circle to its diameter? of its area to the radius? of its circumference to its area?

6. What pairs of numbers in the following series have the same ratio: 2, 3; 5, 7; 1, 4; 10, 15; 3, 12; 14, 21; 25, 35; .25, 1; 1.5, 6; 1.4, 2.1; 3.5, 4.9; .3, 1.2; 3.02, 3.03?

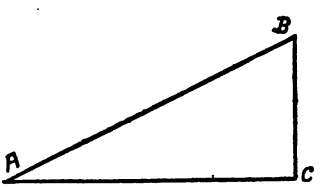
7. What is the present ratio of \$5 to 1 £? \$0.25 to 1 franc? \$0.25 to 1 mark?

8. Show that a per cent as used commercially is a ratio. Illustrate.

9. A man's income is increased 20%. The ratio of his income to his past salary is what? A's cost of living is 120% of what it was 10 yrs. ago; what is the ratio of the present to the past cost of living?

10. Suggest at least 2 recipes for 5 people to be changed to recipes for 2 people; for 8 people.

11. By the gradient or slope of a road is meant *the ratio of the rise to the horizontal distance.* The slope in the accompanying figure is



BC/AC. What is the slope of a road which rises 100′ to the mile? 150′ to the mile?

12. What is the rise in terms of feet of 16 mi. of railroad whose average gradient for that distance is $\frac{1}{70}$? in 20 mi. whose average gradient is $\frac{1}{150}$?

13. Compare 3 : 4 with $\frac{3}{4}$, and 3 : 4 :: C : 8 with $\frac{3}{4} = \frac{C}{8}$.

14. Explain proportion fully. Illustrate.

15. The amount that an employer can afford to pay his employee is proportional to the amount of work the employee turns out in a given time; it is inversely proportional to the time it will take him. Explain each statement fully and illustrate.

16. What should be the ratio of the pay received by A and B if the ratio of the time that it would take each was $\frac{3}{5}$? $\frac{9}{7}$? $\frac{18}{35}$?

17. What is meant by saying that "A is proportional to B?" "A and B are in the ratio of $\frac{4}{7}$?" "A and B are inversely proportional?" "A varies as B?" Illustrate each. Suggest and explain further similar expressions.

18. Prove that the product of the means is equal to the product of the extremes in any proportion.

19. Simple interest on a loan is directly proportional to what elements? Illustrate.

20. What sets of quantities in the graphs of Chapter XXIII were directly proportional? inversely proportional? neither?

21. Specific gravity is the ratio of the weight of a substance to the weight of an equal volume of water. If the weight of water is 62.5 lb. per cubic foot, what is the specific gravity of wood weighing 29 pounds per cubic foot? of copper weighing 557 pounds per cubic foot? of an oak plank 2″ × 12″ × 14′ weighing 122.3 lb.?

22. By the use of the following table find the weight of .4 cu. ft. silver; .2 cu. ft. mercury; a bar of copper 2″ × 2″ × 10″; petroleum in a tank 4′ × 4′ × 8′;

sp. gr. silver	10.5	sp. gr. mercury	13.596
" " copper	8.9	" " petroleum	.7

23. A pair of shoes costing $4.50 could be worn 11 months and a pair costing $2.50 only 6 months. Is the ratio of the prices equal to that of the time worn? Which is the more economical?

24. A pair of shoes costing $4.00 could be worn 10 months while a pair costing $3.00 could be worn 7½ months. Answer the same question as in 23.

25. What number is in the same ratio to 15 that 3 is to 2? to 16 that 2 is to 7? to 17 that 5 is to 6?

26. Divide $240 among A, B, and C so that their sums will be proportional to 3, 4, and 5.

27. Find the height of a tree whose shadow is 40′ when that of a yard measure is 4.5′.

28. What is meant by drawing to scale? The ratio of the length of any line in a drawing to the actual distance is called the scale. What is the scale in the following cases:

A map in which 300 miles are represented by 6 inches.

A baseball diamond represented by a 9″ square; by a $4\frac{1}{2}$″ square; by a 3″ square.

A mechanical drawing in which 14″ is represented by $3\frac{1}{2}$″; 20″ represented by $5\frac{1}{4}$″.

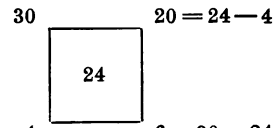
29. Kansas is about 400 miles by 200 miles. What scale should be used to make the best map on a paper 9″ by 12″? 6″ by 9″?

30. In the same way solve the scale to be used for Illinois; for North Carolina.

31. What will be the length in each of problems 29 and 30 of 150 miles? of 50 miles? of 250 miles?

32. Show that a proportion stated in the language of the literal equation is really a shorthand solution by unitary analysis.

33. The following form is used by dairymen in reducing cream: Reduce 60 gallons of 30% cream to 24% cream by adding 4% milk.



$$G = \frac{\overset{3}{\cancel{60}} \times 6}{\underset{2}{\cancel{20}}} = 18 \text{ gallons}$$

Hence to 60 gallons of 30% cream must be added 18 gallons of 4% milk in order to have a 24% cream. Explain the form given.

34. How much 6% milk must be added to 12 gallons of 30% cream to reduce it to 25% cream? How much 20% cream must be added to 10 gallons of 3% milk to raise it to 3.6% milk?

35. State and discuss some of the principal objections to including ratio and proportion in the arithmetic work.

36. How should the subject be taught in order not to make it mechanical?

37. What should be the main object in teaching ratio and proportion?

38. State at least three problems suitable for a grade class in ratio.

39. State at least three problems in proportion suitable for a grade class.

40. Explain each of the terms: majority, plurality, and average.

41. A man sold 23 head of cattle for $1154.64. What was the average price per head?

42. A man paid an average of $35.42 per head for cattle. How many did he buy for $495.88?

43. Criticize Ex. 42 as a good problem. Compare it with 41.

44. Suggest at least three problems involving majorities and averages suitable for a grade class.

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CHAPTER XXV

MENSURATION

264. History. The Ahmes papyrus, the oldest known mathematical document, was written in Egypt between 1700 and 2000 B. C. As it contained information covering a period at least 1000 years previous, it is a record of arithmetical knowledge extending back over 5000 years. Two classes of mensuration problems received special attention at this early time; the volumes of barns in connection with the storing of grain; and the area of plane figures, which was an important part of surveying. The early surveyors, called "rope-stretchers," knew that the triangle whose sides are 3, 4, and 5 is a right triangle even though they had no knowledge of the general relation between the sides of right triangles. The mensuration undertaken by the early Egyptians, Hebrews, Babylonians, Hindus, and Arabs as well as that of the later Romans was of a wholly practical nature. It is to the Greeks, who developed geometry from a few fragmentary facts collected from the Egyptians about 600 B. C., that we owe our present laws of mensuration computations. These studies, which culminated in Euclid's elements, were continued for about 300 years.

Almost every one has heard about the problem of "squaring the circle," which means finding the side of a square whose area equals that of a given circle; in more technical mathematical language it means finding the value of π, or the ratio of the area of a circle to the square of its radius. For many centuries this problem occupied the attention of the great mathematicians of the day.

It may be mentioned in passing, that the Hindus reversed the problem and tried to find the radius of a circle equal in area to a given square. The Hebrews and the Babylonians considered π equal to 3, Ahmes papyrus gives it as $(1\frac{1}{9})^2$, or 3.1605 while ARCHIMEDES placed it between $3\frac{1}{7}$ and $3\frac{10}{71}$. In 1794 LEGENDRE proved it to be irrational; that is, a number which has no ending, like $\sqrt{2}$. It was computed to 35 decimal places by LUDOLPH VON CEULEN and later to 707 decimal places by WILLIAM SHANK. For all purposes in the grades, $3\frac{1}{7}$ is a sufficiently accurate value. By carrying π out to nine decimal places, the circumference of the earth at the equator can be computed accurately to inches. The symbol π was suggested by W. JONES in 1706 and brought into common use by EULER.

The Babylonians divided the circle into 360 degrees, probably due to the fact that they considered the year 360 days long; this is still the standard measurement with the surveyor of today, over 4000 years since it was established.

The student interested in the historical development of the usual formulas or other phases of mensuration must again be directed to the mathematical histories for further information as space will not permit its consideration here.

265. First Four Years. Until the recent movement towards kindergarten and better primary education, no attention was given to mensuration until the last year or two of the grammar school. Children leaving school before the end of the eight years went into the world without any ideas of measurements, of the relations of objects in space, or of lines in a plane. It is not the purpose to repeat here what has previously been offered regarding primary construction work but to emphasize its importance and to suggest to the teacher the necessity of its continuance from year to year, so that there may not be two or three years intervening between this work of the primary and of the

final years. In view of the excellent construction work done in the kindergarten and in the early primary years, one can not offer the difficulties to be encountered as an adequate excuse for delaying the study of mensuration until the last years of arithmetic work. Hence let it proceed throughout the grades.

266. Fundamental Essentials. Mensuration is the numerical, the measuring, side of geometry. It is so closely connected with inventional geometry on the one hand, and denominate numbers on the other, that the teacher must not attempt a separation of the three. Square and cubic measure are both meaningless without applying them to finding areas and volumes of some simple figures. The following are a few general hints for the teacher's direction:

1. Review constantly the construction work and measurements already learned. Add new work from time to time based upon that previously done.

2. A class unacquainted with this fascinating side of school work should be introduced to it at once. The beginning must be quite as simple as it was in the primary grades but progress may be made more rapidly.

3. Do not teach by rule. Lead up to formulas inductively as far as possible. This is the real place for the application of rationalization. It is necessary for the teacher to realize, however, that by the development of formulas we arrive not at rigorous proofs but come rather to logical conclusions. In some cases, it may be well for the teacher to mention that exact proofs may be found in the more exact science of geometry.

4. Make early use of the simple drawing instruments, such as ruler, compass, and protractor.

5. Introduce drawing to scale and the picturing of conditions in problems. The latter devices were previously referred to under the use of object teaching.

6. Avoid the continual use of standard forms of figures—vary them,—especially as to position.

7. Pay special attention to local applications.

8. Place a premium upon neatness and accuracy.

267. Construction Work. Construction work should be carried along with mensuration so that when the pupils are studying the areas of triangles they are also drawing triangles accurately. Except for urging that the teacher carry the construction work along with the study of mensuration, it is impossible to declare any one order of taking up the work as unquestionably the best. Such constructions as drawing perpendicular or parallel lines, or angles of a given size may be taken up as needed in constructing triangles, parallelograms, and so on, or they may precede their application.

268. Inductive Proof of Formulas. The following are mere suggestions for the students to work out in detail who may be teaching texts which do not contain these explanations.

1. *Parallelograms:* A parallelogram ABCD of any convenient size is constructed (construction will give better results) on heavy paper or thin cardboard which is then cut out. Line AE is drawn. Triangle ABE is cut from the

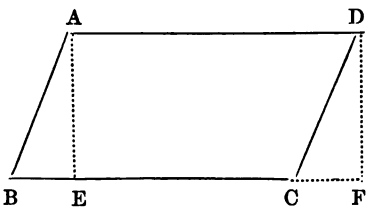


figure and placed in the position DCF. This will result in the rectangle AEFD of the same area as the parallelogram ABCD whose base EF and altitude AE are equal to the base and altitude of the parallelogram ABCD.

2. *Triangle:* Each pupil is directed to construct and cut out two equal parallelograms as was done above. The parallelograms are now divided by cutting, one along the

longer and the other along the shorter, diagonal. This is also the proper place to include the construction of triangles from three given parts. The use of various shaped triangles is advised in order that the pupils may become thoroughly acquainted with the meaning of base and altitude.

3. *Trapezoid:* The formula for the area of trapezoids is easily developed by dividing a trapezoid into two triangles, the formula for which is known.

4. *General Right-Line Figures:* Areas may also be found of some regular figures as pentagons, hexagons, etc., and of figures for which such dimensions are given as will make them divisible into triangles and trapezoids. See rule on page 265.

5. *Circle — Circumference:* Let each child cut out from heavy paper or cardboard, a circle with a radius of his own choosing. Beginning with some mark on the circumference the circle is rolled along a ruler until the mark touches the ruler again. Let each child now divide this measure of the circumference by the length of the diameter he has used, which will give values all close to 3. The teacher may now explain that $3\frac{1}{7}$ is a much closer value for this ratio which the class will thereafter use. The name π and its decimal value 3.1416 should also be used at this time.

6. *Circle — Area:* The division of the circle into equal sectors placed along a straightedge as shown in the usual text is the most common explanation of the formula for the area of a circle. Each sector is a sort of triangle whose area is found by the triangle formula. The total area of the circle is the sum of the areas of the triangles and can be shown to come nearer and nearer the formula $\frac{1}{2}$ radius $\times$ circumference, or $\frac{1}{2} R \times 2 \times \pi \times R = \pi R^2$. The teacher is advised to call attention to the fact that this formula is proved rigorously in geometry. The history of the problem of finding π will also be found very interesting. The inscrib-

ing of circles in and circumscribing them about squares, and *vice versa,* furnishes excellent problems in construction.

7. *Right Triangle:* Take up the study of the relation between the sides of a right triangle before taking up the relations between solid figures. Let each pupil construct a right triangle using any convenient lengths for the two perpendicular sides, as 3, 4; 5, 7; and so on, and then, measure the hypotenuse as accurately as possible. The sum of the squares erected upon the two perpendicular sides will be found in all cases to equal quite closely that erected upon the hypotenuse. It is well for the teacher to point out that inaccuracies in the drawings and inability to measure with sufficient accuracy will give inexact results but that the law can be proved rigorously by geometry. If necessary the law may, of course, be used before this proof is given to the class.

For the complete consideration of the topic, a knowledge of square root is necessary. If square root is to be considered during the same school year there will be no better time to take it up than in connection with this work on the right triangle which is the primary cause for existence of square root in the curriculum. More interest can be stimulated in square root, an otherwise dull subject, by introducing it in connection with the right triangle and thus show the necessity for its place in the curriculum. As a rule the only triangles in the text with integers for sides are those with the sides 3, 4, 5, or some multiple of these. The following scheme for selecting three integers which will form a right triangle is about as simple as any:

$$a^2 + b^2 = c^2 \qquad (1)$$

Hence

$$a^2 = c^2 - b^2 \qquad (2)$$

$$= (c + b)\,(c - b) \qquad (3)$$

Any square may now be selected for a^2 which is factored into two factors $(c + b)$ and $(c - b)$. The sum of the

latter two gives $2c$ and their difference $2b$. Should the factors of a^2 be so selected as to both be even, or odd, $2c$ and $2b$ will both be even and hence b and c will be integers. Suppose 12 is selected for a then the above becomes

$(c+b)\ (c-b)$

(72) (2), or (36) (4), or (18) (8), or (24) (6)

from which $2c = 74, 40, 30, 26$, hence $c = 37, 20, 15, 13$.

$2b = 70, 32, 18, 10$, hence $b = 35, 16, 9, 5$.

Hence there are the following right triangles with all sides integers and one 12: 37, 35, 12: 20, 16, 12: 15, 12, 9: 13, 12, 5.

This permits the construction of circles whose areas equal the joint areas of two or more circles.

8. *Cube and Rectangular Parallelopiped:* The cube and rectangular parallelopiped bear the same relation to solids as the square and rectangle do to plane figures. The method of finding the volumes of the latter is absolutely similar to that of finding the areas of the former. Oblique parallelopipeds arise so seldom that they need not come in for consideration.

9. *Right Prism:* The right prism bears the same relation to the rectangular parallelopiped as the triangle does to the rectangle and should, therefore, be treated in an entirely analogous manner.

10. *Right Cylinder:* In order to find the area of the curved surface of a cylinder wrap a rectangular piece of paper around the cylinder, with one dimension the altitude of the cylinder, and the other a little more than the circumference. Then draw a line along the exposed edge and unwrap the paper from the cylinder. The paper will show the surface of the cylinder in the form of a rectangle whose dimensions are the height and the circumference of the cylinder.

In finding the volume, the cylinder may be treated as a right prism with an unlimited number of sides.

11. *Right Pyramid and Right Cone:* The area of the surface of a pyramid is the sum of the area of the face triangles. If it be a regular pyramid, this area becomes ½ the product of the perimeter of the base and slant height. The area of the surface of a right cone is obtained in the same way, being ½ the product of the slant height and perimeter of the base.

The formula for the volume of a cone is generally arrived at by comparing the amount of water, or sawdust, a hollow tin cone will hold with that of a cylinder of the same base and altitude. Heavy paper can be used for the construction of the cone and cylinder providing they are filled with sawdust, bran, or some other light dry substance. The volume of the pyramid may be found, in a similar fashion, by comparing it with that of a triangular prism of the same base and altitude. Comparisons may also be made of the weight of the sets of solids molded out of clay. In a school having a manual training department a triangular prism may be made and cut into the standard three pyramids of equal bases and altitudes.

The relations between the altitude, the slant height, and the perpendicular distance from the center to a base edge of the pyramid, and that between the altitude, an edge, and the distance from the center to one of the base vertices should receive considerable attention. These relations will be more clearly comprehended if the pupils, themselves, are required to transfer them to a plane. In the case of the cone, edge and slant height become equal.

12. *Sphere:* If a brad or small nail is driven into a hemisphere at its pole, and a cord tied to the nail is wound around on the hemisphere until the surface is entirely covered, it will be found that the same cord is sufficient to

cover the plane circle of the hemisphere twice; hence, the area of the sphere must be four times that of a great circle, or $4\pi R^2$.

The volume of the cylinder whose altitude is $2R$ and radius of whose base R is $2\pi R^3$. The weight of such a cylinder made from putty or clay whose radius is R and that of a sphere of radius R will be found to be in the ratio of $\frac{2}{3}$. Hence, the volume of the sphere will be $\frac{4}{3}\pi R^3$.

13. *Irregular Solids:* Volumes of irregular solids are usually found by immersing them in a vessel of water and noting the amount of water displaced.

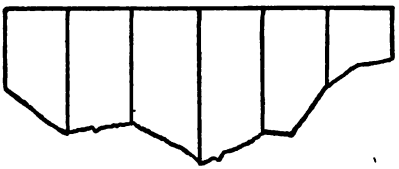
269. Suggestions for Problems. Mensuration presents excellent opportunities for problems along home economics and various industrial lines. The sequel entitled "Problem Material," given below, will be found to contain material for any number of such problems. The actual composition of such problems, however, must be left to the teacher since to offer sufficient variety would take entirely too much space. It is furthermore suggested that this work will gain in the interest that motivation gives if the problems on domestic economy be given to the girls and those upon industrial conditions of the community be given to the boys. In case the boys become sufficiently interested in the problems assigned to the girls to desire to solve these, also, there is no reason why they should not be permitted to do so. Similarly the girls should be permitted to solve those assigned to the boys if they are sufficiently interested.

PROBLEM MATERIAL

1. Cutting out doll's clothes from patterns placed upon cloth (paper may take the place of cloth) in such a way as to make as little waste as possible; cutting out boxes in the same way, etc.

2. Areas of similar figures vary as the square on corresponding sides.

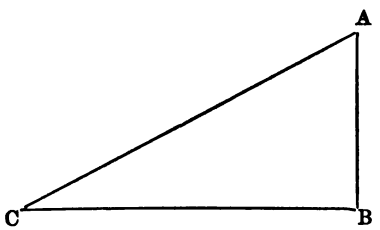
3. The length of time that it takes to play phonograph records varies as the width of the playing part of the record.



4. Irregular areas are often found by dividing the figure into trapezoids, the sum of whose separate areas equals that of the given figure. A rule often used is the following:

Divide the area into an even number of trapezoids as mentioned. Multiply by the length of surface the sum of the first and last vertical, twice that of the even verticals, and four times that of the odd verticals. This divided by three times the number of divisions gives the area.

Excavations and embankments are usually computed by this rule.

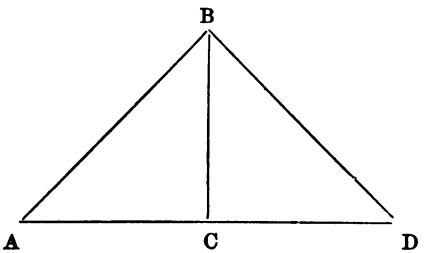


5. A Gunter's chain may be borrowed for the purpose of making real the work in laying out an acre of land and in running perpendicular lines after the manner of surveyors. If a line is desired perpendicular to BC at the point B the surveyor lays out BC equal to 40 links. One end of the chain is held at B, the end of the 80th link is pinned at C; the chain is, then, drawn out taut from the end of the 30th link from B. The line BC, 40 links long, will be perpendicular to the line AB, 30 links long. The proof of this fact is left for the student.

6. In rural communities the cost of farm labor including amount of acreage which a man and team can do per day offers good problems.

7. The architect uses $\frac{1}{4}$ inch in his plans to denote 1 foot in an ordinary building.

8. The drawing of school grounds and of school gardens to scale furnishes excellent problem work on measurements.



9. By pitch of a roof is meant the ratio of BC to AD in the figure.

10. Use local rules and prices in such computations as lathing, plastering, laying cement sidewalks, painting, and paper hanging.

11. One gallon of paint is supposed to cover 250 sq. ft. and its application is considered a good day's work for one man.

12. Volumes of similar figures vary as the cubes of corresponding sides.

13. The surface of any cylinder containing a given volume is least when the depth is twice the radius of the base. This provides excellent material for problems on the construction of cisterns, fruit jars, and so on.

14. Lumbermen use the following formula in approximating the number of board feet in a log:

$$\tfrac{1}{2}\,[\text{smaller end} - 4 \text{ in.}]^2 \times L \div 8 = \text{number of board feet.}$$

L is here given in feet and the smaller end in inches.

XXV. EXERCISES

1. Give a brief summary of mensuration as developed by the ancients.

2. Give a brief summary of the measurements of the circle.

3. Give reports on the development of the formulas for the relation between the sides of the right triangle; area of the triangle; area of the surface of a cone, pyramid, and sphere; volume of a pyramid, cone, and sphere.

4. What use can be made of these historical references in grade work?

5. What should the children, who have completed the primary work in arithmetic, know about mensuration?

6. With what two other topics should mensuration be closely allied? Discuss fully and illustrate.

7. When should mensuration be given in the grade curriculum?

8. Explain how rationalization pertains particularly to mensuration. Illustrate.

9. What is meant by an inductive proof? Illustrate.

10. How rigorous need the proofs of formulas be for grade classes? Illustrate.

11. Explain how the children can be led to prove for themselves the various formulas for areas of plane figures.

12. How many 2-inch squares can be cut from a rectangle 3 by 8 inches?

13. Compare the area of a square with that of an equilateral triangle having the same perimeter.

14. How many square yards of cement will be required for a 5′ side walk inclosing a park 50 by 60 rods?

15. Compare the area of the ground floor of a house with that of an architect's plans.

16. The former is what per cent of the latter? The latter is what per cent of the former?

17. Which is the larger and how much, a garden 20 yards square or of 20 square yards?

18. The former is what per cent of the latter? The latter is what per cent of the former?

19. A trapezoid is 36 sq. ft. in area. If the altitude and one of the parallel sides are 6 and 10 ft. respectively, what is the length of the other side?

20. How would you lead a class to develop inductively the formulas for the circumference and area of a circle?

21. Find the cost of plastering a room 18′ × 13′ × 9′, containing two doors and three windows at ( ) cents per square yard.

Find the cost of papering the above room at 40 cents per roll for material and work.

22. Draw to scale the two elevations of a summer cottage 20′ × 14′ × 8′ with a $\frac{2}{3}$ pitch to the roof. The 2 × 4's are placed 2′ apart and the cottage is sealed only on the outside with drop siding. It is to have a door in an end and a side, and windows on both sides. The floor is to rest on 2 × 6's nailed to the 2 × 4's.

23. Compute the amount of lumber and shingles laid 4″ to the weather which will be necessary to construct the above cottage.

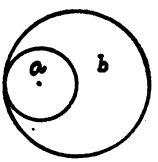
24. Suggest three problems bearing on areas suitable for a grade class.

25. Two 6″ pulleys are placed on parallel journals with their centers 15′ apart. How long a belt is needed to connect the two pulleys?

26. What is the length in miles of a degree of longitude in your latitude? How many miles per hour do you move as a result of the rotation of the earth? feet per second?

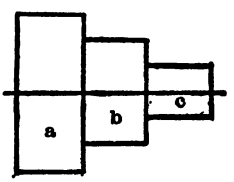
27. How many square yards of surface are there in a cement walk 5′ wide around a circular park containing 5 acres?

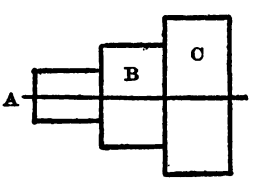
28. How many turns will a 30″ automobile wheel make in traveling a mile?



29. If circle *a* is cut from circle *b*, what will be the area of the remaining part of *b*?

30. A lathe in the manual training rooms and the shaft above has





each a set of pulleys arranged as in the figure whose diameters are 4″, 6″, 8″ respectively. Compare the speed of the lathe with that of

the shaft when the belt passes about pulleys a and A, b and B, and c and C.

31. Suggest three problems on circles suitable for grade classes.

32. How would you lead the children to discover the law governing the relation between the sides of a right triangle?

33. How much farther does the catcher have to throw to second base than to first base?

34. A fielder catches a ball on the right foul line 210 ft. back of first base. How much farther must he throw to third than to second base?

35. Suggest three right triangles whose sides are all integers but not equal to some multiple of 3, 4, 5.

36. Suggest three problems suitable for grade classes in which the triangles made up for Ex. 35 may be used.

37. Give an outline of the construction work you would bring up in connection with mensuration of plane figures. When and how would you take up the construction of perpendiculars, parallels, and angles of a given size?

38. Suggest six problems for construction work on plane figures.

39. How would you lead your pupils to find the area of the surface of a rectangular solid, a pyramid, a cylinder, and a cone?

40. Find the area of the surface of a cone whose altitude is 12″ and radius of the base 5″.

41. Find the entire surface of a pyramid with a square base 8″ and an altitude of 10″.

42. How many thousand gallons are there in a 1″ rainfall over a quarter section?

43. The area of the windows in a well lighted room should equal about 20% that of the floor space. Investigate for your schoolroom.

44. A frustum of a pyramid whose altitude is 3″ has a lower base a 14″ square and the upper base a 6″ square. Find the entire area.

45. A conical tent is to have an altitude of 6′ and radius of its base 4′. Draw an accurate pattern to scale and measure the angle of the sector of the circle used. Will this angle differ as the size of the cone is changed?

46. What angular sector is used in making a conical tent 8′ high and with an 8′ radius? How many yards of tenting will it require?

47. How would you lead the pupils to develop the formulas for the volume of the rectangular solid, the pyramid, the cylinder, and the cone?

48. A pyramid with a square base and an altitude of 18″ has a volume of 294 cu. in. Find the dimension of the base.

49. The volume of a circular cone is 770 cu. in. Find its altitude if the radius of the base is 7″.

50. A cone similar to the one in Ex. 40 has a radius of 10″. Compare their volumes and areas.

51. How much wood must be cut away to turn the largest cylinder from a rectangular block of wood 12″ × 10″ × 8″?

52. A cylindrical glass jar, 10″ high with a radius of 3″ is filled half full of water. A stone dropped into the water raises the level 4″; what is the volume of the stone? How much water would have been forced out of the cylinder if a brick, 2″ × 4″ × 8″, had been immersed in the place of the stone?

53. Find the volume and entire surface of a tomato can of 4″ diameter and 4″ height.

54. Compare the area of the can in Ex. 53 with one of the same volume and 3.5″ high; 4.5″ high. What is your conclusion?

55. Of what practical use is the principle developed in Ex. 54? Apply it to the building of cisterns and reservoirs in general.

56. What should be the depth and radius of the most economical cylindrical cistern holding 60 barrels?

57. How many loads of material were necessary to build up a railroad embankment 350′ long, 9′ high, and upper base 24′ wide and a lower base 52′ wide?

58. Construct from heavy paper a triangular pyramid and prism of equal bases and altitudes.

59. Construct a circular cone and cylinder of equal bases and altitudes.

60. What use can be made of these models in grade work?

61. Outline the construction work you would have your class carry out with the measurements of solids.

62. Suggest two problems suitable for grade work involving each of the regular solids mentioned in Ex. 47.

63. How can formulas for the area of the surface and the volume of a sphere be developed inductively?

64. Compare the areas and volumes of spheres whose radii are 3, 4, and 10 inches respectively.

65. Three clay spheres of radii 1, 2, and 3 inches, respectively, are formed into a single sphere. What is its radius?

66. How much wood is cut off from a wooden 5″ cube in turning out the largest possible sphere?

67. What precautions must always be observed when placing figures in standard positions only? Illustrate.

68. Suggest at least one problem under each of the paragraphs in **269** suitable for grade work.

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CHAPTER XXVI

POWERS AND ROOTS

270. History. Raising numbers to powers and extracting roots are in a sense the fifth and sixth fundamental operations. The earliest existing records are a table of squares of the integers from 1 to 60 and a table of squares of the integers from 1 to 32 found in Babylonian ruins which are about 4000 years old. One book of Euclid is confined to the treatment of irrationals. The Greeks as well as the Hindus were no doubt acquainted with the idea of irrationals long before this time. The relation between the sides of a right triangle, as in the case of the perpendicular sides being 1 and 2, no doubt led to the recognition of the existence of square roots which were irrational. During these early times of course, no systematic process of finding roots was known. As arithmetic is being studied more and more by younger pupils it seems advisable to eliminate the extraction of cube roots of numbers in general.

271. Powers. Very little needs to be said regarding the teaching of powers as the topic is quite simple. If the text used does not contain the exponential notation, as $(15)^2$, it will be a good plan to mention it in the 7th or 8th grade. Special attention should be called to the use of this notation for writing large numbers as 93,000,000 miles, which is written $9.3\ (10)^7$ miles, and to very small numbers as .000003 cm, which is written $3\ (10)^{-6}$ cm.

Several short cuts in involution based upon the algebraic forms $(a+b)^2$ and $(a-b)^2$, etc., are both interesting

and useful. They are used in such cases as $(52)^2$ and $(58)^2$ which become $(50+2)^2$ and $(60-2)^2$. For any number ending in 5 the first becomes

$$(10a+5)^2 = 100a^2 + 100a + 25$$
$$= 100a\,(a+1) + 25.$$

Applying to $(35)^2$, $a = 3$, $a+1 = 4$, hence $100a\,(a+1)$ is $100 \times 3 \times 4 = 1200$. For rapid work it is only necessary to notice that 25 takes the place of the last two zeros and is hence annexed to the product of $a\,(a+1)$.

The number $(352)^2 = (300+50+2)^2$ which, from

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ab+2bc,$$

gives $(352)^2 = (300)^2+(50)^2+(2)^2+2\times300\times50+2\times300\times2+2\times50\times2$

352^2
092504
3020
12
123904

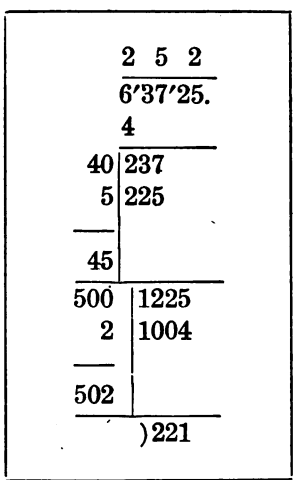
This work can be given in the accompanying form. This form is explained by comparing its figures with those obtained from the above expansion. It has the advantage of not requiring any carrying.

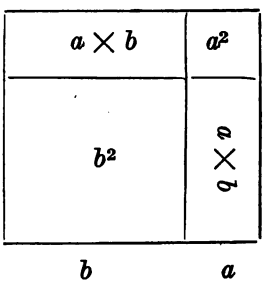
272. Roots. Teach the extraction of roots as the *inverse* of raising to a power. After the pupils are well acquainted with raising numbers to powers, a good introduction of root-extraction may be made by asking incidentally:

What number must be squared to get 36? 25? or what number must be cubed to get 8? 27? and so on. Roots

of simple numbers are now found mentally, for a little time, simultaneously with raising to powers. As the main object of this work is to teach the relation between numbers the examples should take the form of finding the factor and the number of times it is used to produce a given number, as 32 is composed of 2 used as a factor 5 times.

273. Square Root. It is necessary to teach square root because of its repeated use in mensuration. First of all the





pupils should be required to follow minutely the accompanying form, or some other one equally good. Often those who do not follow any form closely are liable to get badly mixed up. The accompanying drawing may be used to explain the form below. Teachers must not, however, insist upon an understanding of this explanation by all the pupils before they are permitted to use the form in actually finding roots.

274. "Carpenters' Rule" for Square Root. The "Carpenters' Rule" can be used in the place of the above form,

which is almost universal. Suppose that the square root of 17 is desired. First, find the number, in this case 4, whose square is less than the given number, 17. Second, divide the given number, 17, by the one just found, 4, which gives $4.25 + ---$. The mean of this last number and the one first found, 4, gives a first approximation of the root, and is 4.125. This number now takes the place of the first and the process is repeated. Thus, $\frac{17}{4.125} = 4.12$, and the mean of 4.125 and 4.12 is 4.1225, or 4.123, which is just the number found by the old process. For $\sqrt{13}$ the first approximation suffices. As 13 is between 3^2 and 4^2 we divide 13 by 3 and get $4.333 ----$. The mean of 3 and $4.33 ---$ is $3.666 ----$, which is about what the old process gives after much more work.

275. Radical Forms that Deserve Study. A certain class of radicals, as $\sqrt{\frac{2}{3}}$, $\sqrt{75}$, and so on, deserve special consideration. All fractions are first to be changed to equivalent ones in which the radical is in the numerator only. This is accomplished by multiplying both terms by a number such that the required root of the denominator can be taken. For instance, $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{\sqrt{15}}{5}$. The square root of 15 is now found in the usual manner and divided by 5.

If the number under the radical has a factor of which the exact root can be taken that is placed outside the radical, as $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$. The square root of 3 is then multiplied by 2. These simplifications give only one root to extract in the case of fractions, give smaller numbers, and often give numbers under the radical, as 2, 3, 6, and so on, for which the required root has been computed to several decimal places and arranged in tables on next page.

	Square Root.		Square Root.
2	1.4142	26	5.0990
3	1.7321	29	5.3852
5	2.2361	30	5.4772
6	2.4495	31	5.5678
7	2.6458	33	5.7446
10	3.1623	34	5.8310
11	3.3166	35	5.9161
13	3.6056	37	6.0828
14	3.7417	38	6.1644
15	3.8730	39	6.2450
17	4.1231	41	6.4031
19	4.3589	42	6.4807
21	4.5826	43	6.5574
22	4.6904	46	6.7823
23	4.7958	47	6.8557

XXVI. EXERCISES

1. What are the six fundamental operations?

2. What are irrational numbers? Illustrate.

3. Why should square root and not cube root be included in the grade arithmetic?

4. Express in simple index notation: 56,000,000, .000,000,56, 34,000, .000,075,2.

5. Express the following without indices: $5.4(10)^5$, $3.2(10)^{-4}$, $6.3(10)^{-3}$, $9.3(10)^7$, $4.2(10)^{-5}$.

6. Give orally the following squares: $(45)^2$, $(35)^2$, $(65)^2$, $(95)^2$, $(125)^2$, $(54)^2$, $(32)^2$, $(48)^2$, $(68)^2$, $(98)^2$.

7. Give the following orally: $\sqrt[5]{32}$, $\sqrt[3]{27}$, $\sqrt{625}$, $\sqrt[3]{125}$, $\sqrt[6]{64}$, $\sqrt[4]{81}$, $\sqrt[5]{243}$, $\sqrt{169}$.

8. Explain how you would introduce the subject of roots as the inverse of raising to a power. Suggest at least six examples.

9. How can involution of small numbers be used to teach number relations? Illustrate.

10. Find to three decimal places: $\sqrt{463.125}$, $\sqrt{45.26}$, $\sqrt{125.48}$, $\sqrt{67.2348}$.

11. Explain as to a class the derivation of the form under 273 from the drawing accompanying it.

12. Find the following values by the "Carpenters' Rule": $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{12}$, $\sqrt{30}$.

Find the following by using the table under 275:

13. $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{5}}$, $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{1}{6}}$, $\sqrt{\frac{1}{8}}$, $\sqrt{\frac{3}{4}}$.

14. $\sqrt{75}$, $\sqrt{12}$, $\sqrt{28}$, $\sqrt{20}$, $\sqrt{640}$, $\sqrt{125}$, $\sqrt{32}$.

15. What is the advantage of simplifying the forms before carrying out the computations?

16. Explain the teaching of the above to a grade class. Suggest at least ten illustrative examples.

17. What percentage of the distance is saved by going along a diagonal in the place of "around" a square? A rectangle twice as long as wide?

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CHAPTER XXVII

GEOGRAPHICAL MEASUREMENTS AND TIME

276. Province of the Study. All questions relating to latitude and longitude, and to the connection between longitude and time are geographical questions requiring an arithmetical treatment; they are quantitative questions for consideration in geography. Because of the arithmetical principles involved, longitude and time is given a chapter in the arithmetic instead of being placed in geography where it naturally belongs. As a successful treatment of these questions must necessarily be from the arithmetical standpoint their discussion for teachers can very properly be taken up here, but their consideration in the elementary schools lies within the province of geography.

277. Introduction of the Subject. In the early grades the children learn that the earth is a huge ball, and that because of its immense size the little part upon which we live, seems to us to be flat. By means of a globe or apple rotated about an axis in the close proximity of a lighted candle, they learn that the rotation of the earth about its axis produces day and night. The equator, parallels, and meridians can be drawn upon the apple or pointed out upon the globe at this time and explained as lines used to locate places upon the earth's surface just as streets are used in a town. Towns having two principal streets from which all the houses are numbered have a ready-made illustration of reference lines very similar to parallels and meridians, in which the two principal streets correspond to

the equator and prime meridian. Maps containing meridians and parallels will help the children to realize that these reference lines are really used in locating places upon the earth's surface. It will be a loss of time and energy to attempt much more than what we have just described before the pupils have attained a knowledge of circles and angular measurements. The study of longitude and latitude in geography can be made an entering wedge in the correlation of arithmetic with geography by preceding it with that of circles and angular measurements in arithmetic.

278. The Fundamental Problems. Three classes of fundamental problems arise: 1, finding the difference in longitude or latitude between two places; 2, finding the difference in time between two places, given their difference in longitude; 3, finding the difference in longitude between two places, given their difference in time. The latter two problems, which are very closely connected, will be considered under a separate paragraph. With the proper prerequisite knowledge of circles and angular measurements this work can very well come during the sixth or seventh grade.

Some sort of globe is necessary for a successful treatment of the above problems. The latitude and longitude of several places should first be determined from the globe and these readings compared with the more exact ones found in the texts. The pupils are now ready to find the difference in latitude or longitude between two places, in finding which, they use the globe for directing them as to "when to add and when to subtract," the only difficulty usually encountered. It often happens that the teacher increases this uncertainty of the pupils by telling them which operation to use for each separate problem instead of approaching the matter through a study of the globe, thus helping the pupils to see for themselves quite easily "when to add and when to subtract." Here, as in all similar instances, the

pupils should never be told *what* to do, but should be directed *how* and *where* to find out the process for themselves. As they grow older, more and more can be left to their discretion and even the references need not be made so specific. Street intersections, as previously mentioned, may be used in clearing up this matter of addition and subtraction. In the country, range and township lines can be used for reference.

272. Longitude and Time. Pupils who can find the difference in latitude and in longitude between two places with facility are ready for the study of the relation existing between longitude and time. A startling, as well as interesting introduction to this study, can be made by asking the class to explain how a newspaper in San Francisco can print at 5 P. M. the account of an event which took place in Boston at 7 P. M. of the same day. The candle and globe previously referred to can again be brought into good use for making clear this strange truth and for explaining the fact that morning, noon, and night occur at different instances in places not on the same meridian. Some little time can be profitably devoted to getting a clear notion of this important fact. An explanation of the change of the seasons is in order at some time during this study, and can easily be demonstrated by moving the globe, inclined at the proper angle, about the candle in a manner similar to that of the earth about the sun.

When the above matters are fully understood, the problem reduces itself to a simple one in evaluation, in which the number of units expressing the difference in time in hours, minutes, and seconds multiplied by the converting factor, 15, gives the difference in longitude expressed in degrees, minutes, and seconds of arc. Time, it must be understood, is not multiplied by 15 giving a product in arc. Hence, the accompanying form of solution is not to be

3 hrs.	12 min.	32 sec.
		15
48°	8′	0″

accepted. This point can, however, be carried too far. Often the pupils are made to feel that such matters when insisted upon at all times are merely hobbies of the teacher. This solution can be accomplished by either multiplication of numbers without names, by 15, and interpretation of results obtained, or, it may be accomplished by the use of the accompanying form, which really amounts to the same thing.

3°	12′	32″
		15
48°	8′	0″

The *inverse* problem is fully as simple. Treated as a *reciprocal process,* we have to undo multiplication in which the multiplier and product are given. The accompanying form will be found a very good one for the solution of the inverse problem.

15) 48 hr.	8 min.	32 sec.
3 hr.	12 min.	34 sec.

Often pupils and even teachers carelessly use the abbreviation for minutes and seconds of arc in making a statement of time and write 3 hr. 6′ 14″ in the place of 3 hr.

6 min. 14 sec. Such carelessness must never be tolerated by the teacher.

280. Standard Time. Standard time, introduced into the United States by the railroads in 1883, is an integral part of business affairs of today. Its advantages over mean solar time are so great that it has been adopted by most of the civilized countries. Throughout Mexico the time of the City of Mexico is used; the British Isles, France, and Spain use Greenwich time; Germany and most of the other European countries use that of the 15th meridian east. The problems arising as a result of this use of standard time are very simple, but the subject deserves prominent consideration because of its importance. A study of this subject must include consideration of the following points:

1. The fundamental principle upon which is based the division of the country into zones resulting from the system of standard time.
2. The name and extent of the four zones of the United States and the one to the east including Labrador, Nova Scotia, and New Foundland.
3. Reason for the irregularity of the zones.
4. The variation in time between zones.
5. Names of some of the principal cities situated within and upon the boundary lines of each zone.
6. Reasons for the establishment of standard time.

281. International Date Line. If a man started from the prime meridian at noon and traveled west at the rate of 15° per hour he would always have noon; and in 24 hours he would be back at the prime meridian, but it would be the *next day*. The query is, when did he change dates? The change in date would be made when he crossed a line which roughly follows the 180th meridian. This line is usually referred to as the *international date* line although strictly speaking there is no such line as it has so far received no international recognition.

XXVII. EXERCISES

1. Define latitude. Give the latitude of three prominent cities in North America and of three in South America.

2. What is the difference in latitude of the cities named in Ex. 1?

3. St. Louis is much farther from San Francisco than from Chicago yet its latitude is much nearer that of the former city. Explain fully the reason for this.

4. Define longitude. Give the longitude of three cities east and of three cities west of London.

5. What is the difference in longitude of the cities named in Ex. 4?

6. When should you "add and when subtract" in finding the difference in longitude or latitude between two places? What use can be made of a globe or map in showing this? Illustrate as to a class.

7. Iceland extends from W. long. 13° 31′ to W. long. 24° 17′ and is 325 mi. long while Cuba extends from W. long. 74° 7′ to W. long. 84° 58′ and is 760 mi. long. Explain the discrepancy.

8. What is meant by "reducing factor?" What is the reducing factor in problems on longitude and time? Explain its use.

9. Explain as to a class the meaning of the relation existing between the difference in longitude and in time of two places.

10. Will noon come earlier or later in New York, Omaha, London, St. Louis, and Minneapolis than in Chicago? How much earlier or later?

11. Consider fully each of the 6 points mentioned in connection with a study of standard time.

12. Explain the international date line. Why does it not follow any one meridian?

13. How will one sail around the earth to "gain" a day? to "lose" a day? Explain fully.

14. Explain the change of seasons by means of a globe and candle.

15. Suggest three problems suitable for class work under each of the fundamental problems mentioned under 278.

16. Suggest three problems suitable for class work on standard time.

17. Suggest an introduction to the subject of longitude and time other than the one mentioned under 279.

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CHAPTER XXVIII

SERIES

282. Arithmetical and Geometrical Series. A few years ago the topics of arithmetical and geometrical series occupied an important position in the advanced grades of arithmetic. Today they are given a place in the appendix or omitted altogether. This is really unfortunate, because conditions are arising which would seem to make an understanding of series useful. In the first place, savings clubs, in which the members deposit 5, 10, 15 cents and so on weekly, are to be found all over the country. The total of these weekly deposits is the sum of an arithmetical series; regular monthly or yearly deposits offer a geometrical series for summation. Buying upon monthly or yearly payments also presents a series for solution.

The arithmetical series is a good illustration of the use of average values. Geometrical series give an insight into the more advanced processes of mathematical reasoning and operations than the pupils can otherwise acquire in arithmetic. Especially does it show the summation of an infinite number of terms when the ratio is less than one, as in the case of

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots\ldots\ldots\ldots$$

It also shows the law governing such changes as the increase of population.

283. Forms of Series. Since many of the present day texts do not include the forms or the application of these series, they are offered here for use by the teacher who may desire to use them.

Arithmetical series:

$$a + [a + d] + [a + 2d] + [a + 3d] + \text{———}$$
$$l = a + [n - l] \times d$$
$$S = \frac{[a + l] \times n}{2}$$

Geometrical series:

$$a + ar + ar^2 + ar^3 + \text{———————}$$
$$l = a \times r^{n-1}$$
$$S = \frac{a^{rn} - a}{r - l}$$

a = first term.
l = last term.
n = number of terms.
S = sum of first n terms.
d = difference between two consecutive terms in an arithmetical series.
r = ratio between any term and the next preceding one in a geometrical series.

Equal payment series:

If a sum, A, is to be paid in n equal, yearly instalments, each equaling p, the rate of interest being r, then p will be found from the formula

$$p = A \frac{r[1 + r]^{n-1}}{[1 + r]^n - 1}$$

If the amounts paid off from the principal each year, that is, less the interest on unpaid portions, are a, b, c, d, ——— k, then

$$p = [b + c + d + \text{———} k]r + a[1 + r]$$
$$p = [c + d + \text{———} k]r + b[1 + r]$$
$$p = [d + \text{———} k]r + c[1 + r]$$
$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$
$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$
$$p = k[1 + r],$$

where k is the last part paid on the principal. From these we get

$$b = a[1 + r]$$
$$c = b[1 + r] = a[1 + r]^2$$
$$d = c[1 + r] = b[1 + r]^2 = a[1 + r]^3$$
and so on,
$$k = a[1 + r]^{n-2}$$

If the first payment is made at the time of making the contract then $A - p$ will equal the sum of the other payments made on the principal, or,

$$A - p = a + b + c + d + \text{———} + k$$
$$= a + a[1 + r] + a[1 + r]^2 + a[1 + r]^3 + \text{——} + a[1 + r]^{n-2}$$
$$= a(1 + [1 + r] + [1 + r]^2 + \text{———————} + [1 + r]^{n-2})$$

The quantities inside the parentheses form a geometrical series so that

$$A - p = \frac{a([1 + r]^{n-1} - 1)}{r}$$

Substituting the value of a thus obtained in

$$p = k[1 + r] = a[1 + r]^{n-1}$$

and solving for p we get

$$p = A\frac{r[1 + r]^{n-1}}{[1 + r]^n - 1}$$

XXVIII. EXERCISES

1. Define and illustrate a mathematical series.
2. Define and illustrate an arithmetical series.
3. Find the sum of:

   (a) the first 12 terms of $3 + 7 + 11 + \text{——}$
   (b) the first 8 terms of $5 + 8 + \text{———}$
   (c) the arithmetical series for ten terms whose first and fifteenth terms are 5 and 47.

4. Define and illustrate a geometrical series.

5. Find the sum of:

(*a*) The first 8 terms of $3 + 6 + 12 + \cdots$

(*b*) the first 6 terms of $5 - 15 + 45 - 135 + \cdots$

(*c*) the first 6 terms of the geometrical series whose first term is 4 and fifth term is 324.

(*d*) the total number of terms of $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$

6. What is the present attitude toward series in the grade arithmetic? Is this justifiable? Why?

7. What ten equal, yearly payments will buy a $1600 tract of land if interest is 6%?

8. Suggest and solve two other problems on the paying of debts by equal instalments.

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